Lecture 17: Online Decision Making Part 1

[[Traditional algs are "one-shot": you're given the input, you produce the output, the end.]]

"online": – inputs/data arrives over time
– decisions must be made on the fly

[[We've seen things like this before in the study of data structures]]

Input: $I = x_1, x_2, x_3, \ldots\ldots$

Alg. must take a decision after each x_i

[[Generally: Impossible for Alg to achieve the optimal solution, because it doesn't have perfect foresight. So we must be content with "approximation algs" — similar to our study of the NP-hard Min-Weight-V.C.]]

def:
- $OPT(I)$ = cost of optimal decisions knowing all of I
- $Alg(I)$ = " " particular online alg "Alg" on I
- Alg's "competitive ratio on I": $\dfrac{Alg(I)}{Opt(I)}$ (smaller is better)
- Alg's "competitive ratio": $\max\limits_{I} \left\{ \dfrac{Alg(I)}{Opt(I)} \right\}$ [[Worst factor Alg suffers versus perfect knowledge of I]]

[[The most classic of all examples...]] "Ski rental problem"

[[You've never skied, don't know how much you'll like it. You could buy expensive skis now, but maybe you'll hardly use them. Or, could rent every day, but maybe you'll go zillions of times...]]

[[Also like the "wait for the elevator or take the stairs" problem.]]

Cost to rent: \$50.
To buy: \$500.

Input: ski, ski, ski, ..., quit [So, lifetime, you'll go some unknown # of days.]

Alg 1: Buy on 1st day.

$I = 20$ days → $Opt(I) = \underline{500}$, $Alg(I) = \underline{500}$,
C.R. on I: 1 ☺

$I = 4$ days → $Opt(I) = 200$, $Alg(I) = 500$
C.R. on I: 2.5 ☹

$I = 1$ day → $Opt(I) = 50$, $Alg(I) = 500$
C.R. on I: 10 ☹

$\boxed{C.R. = 10}$ ☹

Alg 2: Always rent.

$I = 4$ days → $Opt(I) = 200$, $Alg(I) = 200$ ☺
C.R. on $I = 1$

$I = 20$ days → $Opt(I) = 500$, $Alg(I) = 1000$,
C.R. on $I = 2$ ☹

$I = 1000$ days → $Opt = 500$, $Alg = 50{,}000$
C.R. on $I = 100$

$\boxed{C.R = \infty}$

Alg 3: "BLTN": Better Late Than Never: Rent the first 9 times, then buy on 10th.

Q: For this alg, which I is worst?

A: $I = 10$ $\left(= \frac{500}{50}\right)$

$Alg(I) = 950$, $Opt(I) = 500$

$\boxed{C.R. = \frac{950}{500} = 1.9}$

[finite!]

Generally: Rent cost 1, Buy cost B [[say B is an integer]]

BLTN's C.R. is worst on input $I = B-1$

$Alg(B-1) = 2B-1, \quad Opt(B-1) = B$

$$\Rightarrow \boxed{C.R. = \frac{2B-1}{B} = 2 - \frac{1}{B}} \quad [[< 2]]$$

Fact: This is best (smallest) possible C.R. for any (deterministic) Alg!

Proof: Any (deterministic) alg. is just "rent $T-1$ times, then buy" (on T^{th}) for some $1 \le T \le \infty$ [["∞" means "always rent"]].

(BLTN is $T = \lfloor B \rfloor$.)

Consider $I = T$. [[Intuitively, worst # of days.]]

$Alg(T) = T - 1 + B$

$Opt(T) = \,?$ [[Depends how it compares to B.]]

Case 1: $T \geq B$

$\Rightarrow Opt(T) = B$. C.R. on $I = T$ is $\frac{T-1+B}{B}$

$$\geq \frac{B-1+B}{B} = 2 - \frac{1}{B}.$$

Case 2: $T \leq B-1$

$\Rightarrow Opt(T) = T$ C.R. on $I=T$ is ...

$$\frac{T-1+B}{T} = 1 + \frac{B-1}{T} \geq 1 + \frac{B-1}{B-1} = 2.$$

Either way, $\boxed{\text{C.R. of Alg} \geq 2 - \frac{1}{B}.}$ ∎

[[Why did I repeatedly emphasize "deterministic"?]]

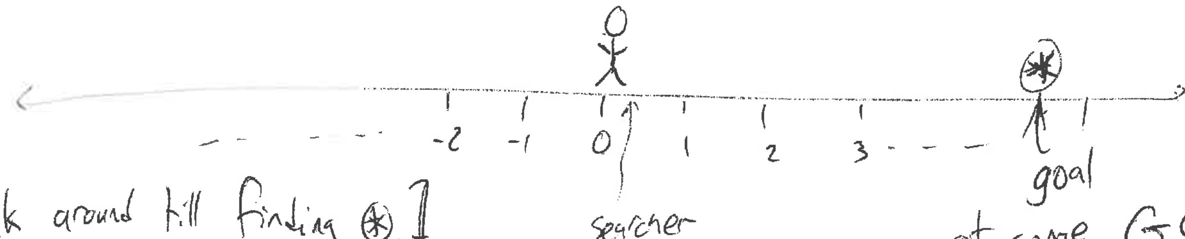
Fact: [[Won't prove.]] $\exists$ randomized alg. with competitive ratio

$$C.R. = \max_I \frac{\mathbb{E}[Alg(I)]}{Opt(I)} = \frac{e}{e-1} \approx 1.58$$

[[And this is optimal!]]

Another e.g.: <u>Hide & Seek</u> (Optimal Search)

[Often studied on geometric graphs but we'll just consider the number line.]



at some $G \in \mathbb{Z}$.

[You must walk around till finding ⊛.]

[like escaping a linear desert]

Opt $= |G|$.

Alg idea? Go to 1, then -2, then $3, -4, 5, \ldots$

But say G large, odd, positive (e.g. 99)

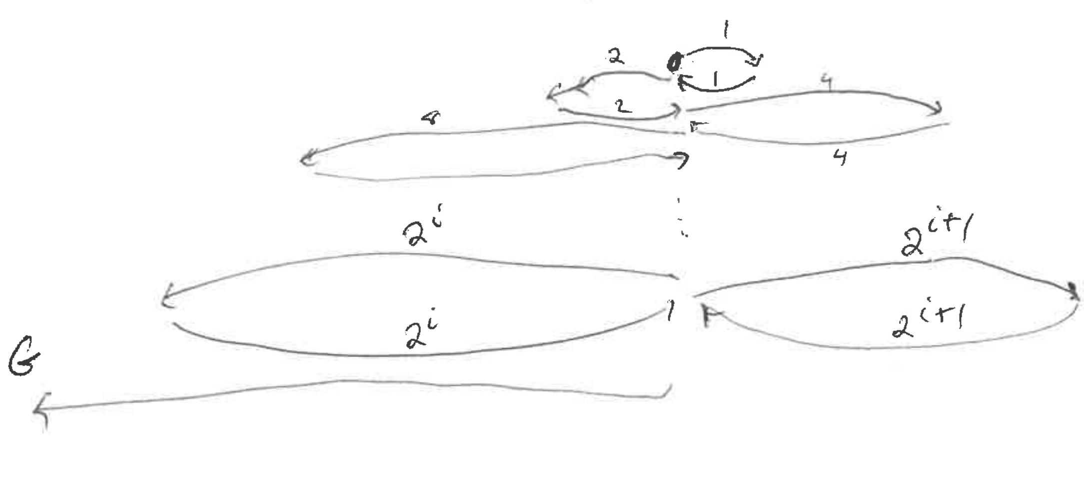
$$\text{Alg} = 1 + 3 + 5 + 7 + 9 + \cdots (2G-1)$$
$$= G^2$$

C.R of this alg on this inst.: $|G|$

$\therefore$ C.R. $= \infty$. ☹

Better alg: "Doubling search":

Go to $1, -2, +4, -8, +16, \ldots$



Suppose G is negative. // G positive has basically same analysis; I leave you to check it.

Say $G < -2^i$, $G \geq -2^{i+2}$.

$\therefore$ Alg's cost is $2(1+2+4+8+\ldots+2^{i+1}) + |G|$

$= 2(2^{i+2}-1) + |G|$

$\leq 2 \cdot 2^{i+2} + |G| = 8 \cdot 2^i + |G|$.

(and $\approx$)

Opt $= |G|$

$\therefore$ C.R. on G $\leq \dfrac{|G| + 8 \cdot 2^i}{|G|} = 1 + 8 \cdot \dfrac{2^i}{|G|} \leq 1 + 8 = 9$.

$(|G| \geq 2^i)$

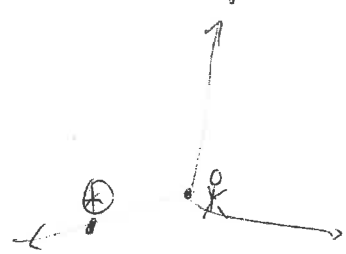
$\boxed{\therefore \text{ C.R} \leq 9}$

thm [not so easy]: 9 is best possible for deterministic algs

fact: $2e \approx 5.4$ is possible (& best) for randomized algs.

Generalization:

3 legs:



fact: Best (det.) strategy:

- 1 on path 1 (then back)
- ≈ 1.5 " " 2 " "
- $(1.5)^2$ " " 3 " "
- $(1.5)^3$ " " 1 " "

etc.

Ex: C.R. ≤ 14.5

L legs: exponential incr. search by factor $\frac{L}{L-1}$.

$$\text{C.R} = 1 + 2L\underbrace{\left(\tfrac{L}{L-1}\right)}^{L-1}. \quad (??)$$

$1 + \frac{1}{L-1} \approx e^{\frac{1}{L-1}}$ $\quad \therefore \approx 1 + 2L \cdot \left(e^{\frac{1}{L-1}}\right)^{L-1} = 1 + 2eL \approx 5.4L = O(L)$

[[Application: tasks where there are several options, liner cost to switch options