# Lecture 25: Spectral Graph Theory II

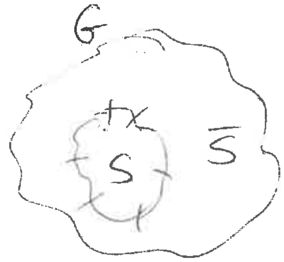
Let $G \overset{=(V,E)}{}$ be a (regular) undirected graph.

Let $S \subseteq V$.

"Conductance" of $S$: $\Phi[S] = \Pr_{(u,v) \in E}[\, v \notin S \mid u \in S \,]. \in [0,1]$.

("escape probability")

[Pick a random vtx in S. Take one random step, see if you're outside S.]




Dumb: $\Phi[V] = 0$.

Focus on smaller of $S, \bar{S}$ $\iff$ $|S| \le \frac{n}{2}$.

$\Phi[S] \le$ small $\longrightarrow$ bottleneck for random walk.

think $\approx \frac{1}{(\log n)^c}$

$\Phi[S]$ not small $\forall S$ ($|S| \le \frac{n}{2}$) $\longrightarrow$ no bottlenecks $\rightsquigarrow$ rand walk converges quickly

$\rightsquigarrow$ diameter is small ($\le (\log n)^c$).

**Key idea** [underlying many of the biggest developments in Algorithms in last few years].

Every graph $G$ looks like...


$G_i = (V_i, E_i)$

"Expander decomposition"

- each $G_i$ has no bottlenecks: $\Phi[S] \ge \frac{1}{(\log n)^c}$ $\forall S \subseteq V_i$ $\left(|S| \le \frac{|V_i|}{2}\right)$

  [implies polylog diam, $G_i$ is statistically similar to a rand graph]

- only $\frac{1}{(\log n)^c}$ fraction of edges of $G$ go between diff. $G_i$'s

  [so mostly ignorable]

• You can <u>find</u> this decomposition in <u>near-linear</u> time ②

$O(m \cdot (\log n)^c)$.

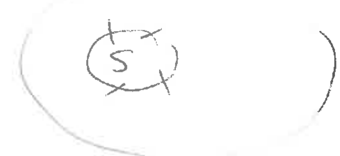
["near linear"]

[Saranurak-Wang '21]
[best params]

→ Crucial ingred. in... near-linear time Max-Flow alg [Chen, Kyng, Liu, <u>Peng</u>, Gutenberg, Sachdeva '22]

→ All-pairs Max-Flow / min-st-cut in undirected graphs in near-linear time [Abboud, Krauthgamer, <u>Li</u>, Panigrahi, Saranurak, Trabelsi '22]

Basic idea: Given $G$, find set $S \subseteq V$ with smallest(ish) $\Phi[S]$. If it's large → done.
Else partition off, recurse.
[few edges on its boundary].



<u>Today</u>: how to do this efficiently?

$$\Phi[S] = \Pr_{(u,v)\in E}[v \notin S \mid u \in S] = \Pr_{(u,v)\in E}[u \in S, v \notin S] \Big/ \Pr_{(u,v)\in E}[u \in S] \quad \Leftrightarrow u \in V$$

$$= \mathcal{E}[f_S] / \langle f_S, f_S \rangle$$

for $f_S : V \to \mathbb{R}$ defined by $f_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases}$

<u>Recall</u>: • $\langle g,h \rangle = \mathop{\mathbb{E}}_{v\in V}[g(v)h(v)]$, so $\langle f_S, f_S \rangle = \mathop{\mathbb{E}}_{v \in V}[f_S(v)^2] = \Pr_{v\in V}[v \in S]$

$f_S(v)$

• $\mathcal{E}[g] := \frac{1}{2} \mathop{\mathbb{E}}_{(u,v)\text{ edge}}\left[(g(u)-g(v))^2\right] = \langle f, Lf \rangle$,

where $L = I - K = I - \frac{1}{d}A$.

("Laplacian")

$K$ is a symmetric matrix $\Rightarrow L$ is too ③

$\Rightarrow$ There are orthonormal (eigenvectors

"normalized": $\langle \varphi_i, \varphi_i \rangle = \mathbb{E}_v[\varphi_i(v)^2] = 1$

$\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ (recall: vector $\equiv$ function: $V \to \mathbb{R}$)

& scale factors (eigenvals) $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$

such that $L\varphi_i = \lambda_i \cdot \varphi_i$.

So $\mathcal{E}[\varphi_i] = \langle \varphi_i, L\varphi_i \rangle = \langle \varphi_i, \lambda_i \cdot \varphi_i \rangle = \lambda_i \langle \varphi_i, \varphi_i \rangle$
$= \lambda_i$.

[[Idea: $\varphi_0, \ldots, \varphi_{n-1}$ are a basis for <u>all</u> $f: V \to \mathbb{R}$, and we know their "energies" $\lambda_i$]]

Recall: $\mathcal{E} \geq 0$ always, $\leq 2$ for normalized $\varphi$.

$\therefore$ [after ordering $\varphi_i$'s by $\lambda_i$...] $0 \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq 2$

= always

$\varphi_0 \equiv 1$.

equal if $G$ bipartite. (recall)

recall: $L$ has $k$ eigs $= 0$
$\Leftrightarrow$ $G$ has $k$ connected comps.

[[Incidentally, eigs of $K$ are almost the same...]]

$L\varphi_i = \lambda_i \varphi_i \Leftrightarrow I\varphi_i - L\varphi_i = \varphi_i - \lambda_i \varphi_i$

$\Leftrightarrow K\varphi_i = (1-\lambda_i)\varphi_i$

($L = I - K$, $K = I - L$)

$\therefore$ $\varphi_i$'s also the eigvecs of $K$,

eigval. $\kappa_i := 1 - \lambda_i$. $-1 \leq \kappa_{n-1} \leq \ldots \leq \kappa_1 \leq \kappa_0 = 1$

eq. if bipartite.

[largest eigval: $\varphi_0 \equiv 1$

Recall (Lec. 10) Power Method [[to find eigenvector of a symm. matrix of largest (absolute) eigenval]]

Start from any vector $\pi_0$ (with at least slight overlap with largest eigenvec. $\varphi_0$)

④

[We chose a random $\Pi_0$ to make sure, whp, it had slight overlap with largest eigenvec. Here, we <u>know</u> largest eigenvec is $\varphi_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, so anything with positive numbers OK.]

Repeatedly apply $K$ → converges to $\varphi_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$ (up to scaling)

If $\Pi_0$ is <u>any</u> $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, get very close to after $t = O\left(\frac{\log n}{\varepsilon}\right)$ steps

$\varepsilon$: gap between 1st & 2nd largest eigenvals

But $K^t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow u$ is prob. distribution after doing $t$-step random walk from $u$!

$\therefore$ if $|k_{n-1}|, \ldots, |k_1| \le 1-\varepsilon$, $O\left(\frac{\log n}{\varepsilon}\right)$ walk steps get you from $u$ to almost-uniformly distributed!

(e.g.: $\varepsilon = \frac{1}{(\log n)^c}$ ⇒ polylog(n) steps ⇒ <u>diameter</u> $\le$ polylog(n) !)

"G is an "<u>$\varepsilon$-(spectral)-expander</u>"".

↳ Not true if, e.g. $k_{n-1} = -1$ ≡ bipartite. [We knew that... random walk doesn't converge...]

↳ Simple fix: In random walk, stay still each step with prob. 1/2. ["lazy walk"; fixes odd-even dumbness]

↳ $K \to \begin{bmatrix} 1/2 & & 0 \\ & 1/2 & \\ 0 & & \ddots \; 1/2 \end{bmatrix} + \frac{1}{2}K = \underbrace{\frac{1}{2}I + \frac{1}{2}K}$

eigenvalues are $\underbrace{\frac{1}{2} + \frac{1}{2}k_i}$, same eigenvecs.

↓

between 0 & 1 :)

[no negative eigvals]

$k_1 = 1-\varepsilon \rightsquigarrow \frac{1}{2} + \frac{1}{2}(1-\varepsilon) = 1 - \frac{1}{2}\varepsilon$

gap smaller only by factor 1/2.

[Intuitively, then:] $\kappa_1$ near $1 \Leftrightarrow \lambda_1$ near $0$ $\overset{?}{\Leftrightarrow}$ $\exists$ bottleneck

[= cluster you can peel off in expander decomp.]

<u>Prop:</u> If $\exists S \subseteq V$, $|S| \le \frac{n}{2}$ with $\Phi[S] \le \delta$,

then $\kappa_1 \ge 1 - 2\delta \Leftrightarrow \lambda_1 \le 2\delta$.

<u>Proof:</u> Recall $\lambda_0 = 0$ because for $\varphi_0 \equiv 1$, $\mathcal{E}[\varphi_0] = \langle \varphi_0, L\varphi_0 \rangle = 0 \cdot \langle \varphi_0, \varphi_0 \rangle$

Must turn $S$ into some $f$, orthogonal to $\varphi_0$, with

$\mathcal{E}[f] = \langle f, Lf \rangle \le 2\delta \cdot \langle f, f \rangle$.

$\langle f, \varphi_0 \rangle = 0$

$\mathbb{E}_v[f(v) \cdot \varphi_0(v)] = 0$

average val. of $f$ is $0$.

Idea 1: $f = f_S$; $f_S(v) = \begin{cases} 1 \text{ if } v \in S \\ 0 \text{ if } v \notin S \end{cases}$

Doesn't have $\mathbb{E}[f_S(v)] = 0$, but does have

$\delta \ge \Phi[S] = \mathcal{E}[f_S] / \langle f_S, f_S \rangle$ $\quad \mathbb{E}_v[f_S(v)^2] = \mathbb{E}[f_S(v)] = \frac{|S|}{|V|} =: \mu \le \frac{1}{2}$.

$(\mathcal{E}[f_S] \le \delta \cdot \langle f_S, f_S \rangle)$ ⊛

<u>Fix:</u> Let $f_S'$ be defined by $f_S'(v) = f_S(v) - \mu$.

By construction, $\mathbb{E}_v[f_S'(v)] = \mathbb{E}[f_S(v)] - \mu = 0$

$$\mathcal{E}[f_S'] = \frac{1}{2} \mathbb{E}_{(u,v) \in E}\left[(f_S'(u) - f_S'(v))^2\right] = \frac{1}{2} \mathbb{E}_{(u,v) \text{ edge}}\left[(f_S(u) - f_S(v))^2\right] = \mathcal{E}[f_S].$$

$$\langle f_S', f_S' \rangle = \langle f_S - \mu, f_S - \mu \rangle = \mathbb{E}_v\left[(f_S(v) - \mu)^2\right] = \mathbb{E}[f_S(v)^2] - 2\mu \mathbb{E}[f_S(v)] + \mu^2$$

$$= \mu - 2\mu^2 + \mu^2 = \mu - \mu^2$$

$\approx \mu$, $\because \mu \le \frac{1}{2}$

$\frac{1}{2}\mu = \mu - \frac{1}{2}\mu \le \cdot \le \mu$

$$\therefore \mathcal{E}[f_S'] = \mathcal{E}[f_S] \le \delta \langle f_S, f_S \rangle = \delta \cdot \mu \le \underbrace{\delta \cdot 2}_{2\delta} \langle f_S', f_S' \rangle. \quad \square$$

$\therefore$ if $G$ has <u>no</u> "bottleneck" ($S$ with $\Phi[S] \le \delta$), then $\kappa_1 \le 1 - 2\delta$ $(\lambda_1 \ge 2\delta)$

But what about converse? $\kappa_1 \ge 1 - 2\delta$. Say you find eigenvec $\varphi_1$ (orthog to $\varphi_0 \equiv 1$) with eigenvalue $\ge 1 - 2\delta$, using Power Method.

But... $\varphi_1$ could have <u>any</u> values; not nec. 1 (or $1-\mu$) on some $S$, 0 (or $-\mu$) on $\overline{S}$.... 

"<u>Cheeger's Ineq.</u>": Say $\varphi_1 : V \to \mathbb{R}$ has eigenvalue $1-2\delta$ for $K$.

[[takes about 30 mins to prove]]

Then for some cutoff $c \in \mathbb{R}$ [[try "all" vals]],

if you define

$$S = \{v : \varphi_1(v) \geq c\},$$

then $\Phi[S] \leq O(\sqrt{\delta})$

a bit bigger than $\delta$, but still $\frac{1}{(\log n)^c}$ if $\delta = \frac{1}{(\log n)^c}$.

So: Find $\varphi_1$, $\lambda_1$. If $\lambda_1 \leq 1-\delta \to$ ☺ "expander".

Else find cutoff $c$, set $S \subseteq V$ with $\Phi[S] \leq O(\sqrt{\delta})$

peel off this cluster, recurse.