Adequacy of Regular Expression Encoding (Forward Direction)

Stephen Magill

April 21, 2004

Theorem. *If* $x \in L(r)$ *then* $\Gamma_r(s, f); \cdot \Vdash \forall y.\ s(x \cdot y) \multimap f(y)$.

Proof. We proceed by structural induction on r.

Case $r = r_1 \cap r_2$:

By our construction, $\Gamma_r(s, f) =$
$\forall x.\ s(x) \multimap s_1(x) \otimes s_2(x)$
$\Gamma_{r_1}(s_1, f_1)$
$\Gamma_{r_2}(s_2, f_2)$
$\forall x.\ f_1(x) \otimes f_2(x) \multimap f(x)$

By our inductive hypothesis,

$$\text{If } x_1 \in L(r_1) \text{ then } \Gamma_{r_1}(s_1, f_1); \cdot \Vdash \forall y.\ s_1(x_1 \cdot y) \multimap f_1(y).$$

and

$$\text{If } x_2 \in L(r_2) \text{ then } \Gamma_{r_2}(s_2, f_2); \cdot \Vdash \forall y.\ s_2(x_2 \cdot y) \multimap f_2(y).$$

We will use these facts to construct a proof that if $x \in L(r)$ then $\Gamma_r(s, f); \cdot \Vdash \forall y.\ s(x \cdot y) \multimap f(y)$.

Assume $x \in L(r)$. Since $r = r_1 \cap r_2$, we have that $x \in L(r_1) \cap L(r_2)$ and thus $x \in L(r_1)$ and $x \in L(r_2)$. This satisfies the conditions in the inductive hypotheses and thus gives us the following facts:

$$\Gamma_{r_1}(s_1, f_1); \cdot \Vdash \forall y.\ s_1(x \cdot y) \multimap f_1(y)$$

$$\Gamma_{r_2}(s_2, f_2); \cdot \Vdash \forall y.\ s_2(x \cdot y) \multimap f_2(y)$$

We can apply weakening (which holds for the unrestricted context) to get

$$\Gamma_r(s,f);\cdot \quad \Vdash \quad \forall y.\ s_1(x\cdot y) \multimap f_1(y) \tag{1}$$
$$\Gamma_r(s,f);\cdot \quad \Vdash \quad \forall y.\ s_2(x\cdot y) \multimap f_2(y) \tag{2}$$

Now we must prove $\Gamma_r(s,f);\cdot \Vdash \forall y.\ s(x\cdot y) \multimap f(y)$.

And here is a proof:

$$\dfrac{\dfrac{\dfrac{}{\Gamma_r(s,f);\cdot \Vdash \forall x.\ f_1(x)\otimes f_2(x) \multimap f(x)}\,Hyp}{\Gamma_r(s,f);\cdot \Vdash f_1(y')\otimes f_2(y') \multimap f(y')}\,\forall E \qquad \begin{array}{c}\vdots\\ \Gamma_r(s,f);s(x\cdot y') \Vdash f_1(y')\otimes f_2(y')\end{array}}{\dfrac{\dfrac{\Gamma_r(s,f);s(x\cdot y') \Vdash f(y')}{\Gamma_r(s,f);\cdot \Vdash s(x\cdot y') \multimap f(y')}\multimap I}{\Gamma_r(s,f);\cdot \Vdash \forall y.\ s(x\cdot y) \multimap f(y)}\,\forall I^{y'}}\multimap E$$

This is where things started getting less clear in recitation. It's also where the proof tree starts getting really wide, so I will switch to our linear notation. Let's examine that top-right subgoal on its own.

	First, we show that you can get s_1 and s_2 from s.	
1.	$\Gamma_r(s,f);s(x\cdot y') \Vdash s(x\cdot y')$	[Lin Hyp]
2.	$\Gamma_r(s,f);\cdot \Vdash \forall x.\ s(x) \multimap s_1(x)\otimes s_2(x)$	[Hyp]
3.	$\Gamma_r(s,f);\cdot \Vdash s(x\cdot y') \multimap s_1(x\cdot y')\otimes s_2(x\cdot y')$	[∀ E 2]
4.	$\Gamma_r(s,f);s(x\cdot y') \Vdash s_1(x\cdot y')\otimes s_2(x\cdot y')$	[⊸ E 3 1]
	Next, we get the first I.H. into a useful form	
5.	$\Gamma_r(s,f);\cdot \Vdash \forall y.\ s_1(x\cdot y) \multimap f_1(y)$	[I.H.]
6.	$\Gamma_r(s,f);\cdot \Vdash s_1(x\cdot y') \multimap f_1(y')$	[∀ E 5]
7.	$\Gamma_r(s,f);s_1(x\cdot y') \Vdash s_1(x\cdot y')$	[Lin Hyp]
8.	$\Gamma_r(s,f);s_1(x\cdot y') \Vdash f_1(y')$	[⊸ E 6 7]
	Same for the second I.H.	
9.	$\Gamma_r(s,f);\cdot \Vdash \forall y.\ s_2(x\cdot y) \multimap f_2(y)$	[I.H.]
10.	$\Gamma_r(s,f);\cdot \Vdash s_2(x\cdot y') \multimap f_2(y')$	[∀ E 9]
11.	$\Gamma_r(s,f);s_2(x\cdot y') \Vdash s_2(x\cdot y')$	[Lin Hyp]
12.	$\Gamma_r(s,f);s_2(x\cdot y') \Vdash f_2(y')$	[⊸ E 10 11]
	We combine the two I.H.s	
13.	$\Gamma_r(s,f);s_1(x\cdot y'),s_2(x\cdot y') \Vdash f_1(y')\otimes f_2(y')$	[⊗ I 8 12]
	And finally, we use ⊗-elim on the statement we proved in line 4 and the combined I.H.s	
14.	$\Gamma_r(s,f);s(x\cdot y') \Vdash f_1(y')\otimes f_2(y')$	[⊗ E 4 13]

□