15-399 Supplementary Notes: Double Negation Elimination

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1 Proof of the Gödel-Gentzen Embedding

The Gödel-Gentzen embedding P^* of classical into constructive logic is defined by induction on the structure of propositions as follows:[1]

$$\begin{array}{rcl} P^* & = & \neg\neg P \quad (P\ \mathit{atomic}) \\ \bot^* & = & \bot \\ \top^* & = & \top \\ (P \wedge Q)^* & = & P^* \wedge Q^* \\ (P \vee Q)^* & = & \neg(\neg P^* \wedge \neg Q^*) \\ (P \supset Q)^* & = & P^* \supset Q^* \end{array}$$

This translation expresses the constructive content of classical logic. In classical logic we always have the option of proving a proposition by contradiction (proving $\neg\neg P$, but stating it as a proof of P). Classical logic is also weaker when it comes to disjunction: rather than prove one or the other disjunct, we may instead prove that both cannot fail to hold true. From a constructivist viewpoint the classical proof proves less than it claims. The Gödel-Gentzen translation makes this precise.

From a classical point of view the translation does nothing.

Theorem 1.1 *Classically, P and P^* are equivalent.*

Proof: Proceed by induction on the structure of P, using truth tables to show the equivalence. □

From a constructive viewpoint, it "constructivizes" classical logic. We write $\vdash_{class} P\ \mathit{true}$ to mean that $P\ \mathit{true}$ is derivable using the rules of constructive logic, plus the law of the excluded middle ($P \vee \neg P\ \mathit{true}$ for every P).

Theorem 1.2 *If $\vdash_{class} P\ \mathit{true}$, then $\vdash P^*\ \mathit{true}$.*

[1]The course notes write P^N instead of P^*.

This means that the constructivist may always reinterpret what the classicist says in constructive terms.

Our aim is to give a proof of this theorem. The proof is a bit tricky in spots, which is why we outline it here.

A proposition P is *stable* iff $\neg\neg P \supset P$ *true*. Recall that it is easy to show constructively that $P \supset \neg\neg P$ *true*.

Lemma 1.1 *Every negated proposition is stable. That is, if* $P = \neg Q$*, then* $\neg\neg P \supset P$ *true.*

Proof: Assume $\neg\neg P$ *true*, that is $\neg\neg\neg Q$ *true*. We are to show P *true*, that is $\neg Q$ *true*. Assume towards a contradiction that Q *true*. It follows that $\neg\neg Q$ *true*. But this contradicts the assumption $\neg\neg\neg Q$ *true*. □

An implication is stable if its consequent is stable.

Lemma 1.2 *If* R *is stable, then* $Q \supset R$ *is stable.*

Proof: Assume $\neg\neg(Q \supset R)$ *true*. To show $Q \supset R$ *true*, assume Q *true*. Since R is stable, it is enough to show $\neg\neg R$ *true*. So assume towards a contradiction that $\neg R$ *true*. We will show $\neg(Q \supset R)$ *true* to obtain a contradiction. So assume $Q \supset R$ *true*. Then since Q *true*, we have R *true*, which contradicts the assumption that $\neg R$ *true*. □

Lemma 1.3 *For all propositions* P*,* P^* *is stable.*

Proof: By induction on the structure of P.

1. If P is atomic, then $P^* = \neg\neg P$, so it is negated. By the Lemma 1.1, it is stable.
2. If P is $\bot$ or $\top$, it is easy to show stability.
3. If $P = Q \wedge R$, the result follows by induction, using the fact that $(P \wedge Q)^* = P^* \wedge Q^*$.
4. If $P = Q \vee R$, then P^* is negated, and hence is stable by Lemma 1.1.
5. If $P = Q \supset R$, then by induction R is stable, and hence P is stable by Lemma 1.2.

□

If $\Gamma = P_1$ *true*$, \ldots, P_n$ *true*, then $\Gamma^* = P_1^*$ *true*$, \ldots, P_n^*$ *true*.

Theorem 1.3 *If* $\Gamma \vdash_{class} P$ *true, then* $\Gamma^* \vdash P^*$ *true.*

Proof: By induction on the derivation of the assumption.

1. Suppose that $P = Q \vee \neg Q$ and that we have derived P *true* by applying the law of the excluded middle. We are to show that $\neg(\neg Q^* \wedge \neg\neg Q^*)$ *true*. So assume $\neg Q^* \wedge \neg\neg Q^*$ *true*. But this is a contradiction!

2. Suppose that $P = Q \supset R$, and that we derived P *true* by implication introduction, assuming Q *true* and deriving R *true*. Then by induction we have proved constructively R^* *true* from the assumption Q^* *true*, and hence $Q^* \supset R^*$ *true*. That is, $(Q \supset R)^*$ *true*.

3. Suppose that we have derived P *true* from $Q \supset P$ *true* and Q *true*. By induction $Q^* \supset P^*$ *true* and Q^* *true*, so P^* *true*, as required.

4. Suppose that $P = Q \vee R$ and that we have derived P *true* by $\vee$-introduction (left) from Q *true*. By induction Q^* *true*, and hence P^* *true*. The symmetric case is handled similarly.

5. Suppose that P *true* is derived from $Q \vee R$ *true*, P *true* assuming Q *true*, and P *true* assuming R *true*, using $\vee$-elimination. By induction $\neg(\neg Q^* \wedge \neg R^*)$ *true* is derivable constructively. Moreover, P^* *true* is derivable constructively from Q^* *true* and also from R^* *true*. We are to show P^* *true*. By Lemma 1.3 it is enough to show $\neg\neg P^*$ *true*. So assume $\neg P^*$ *true*, and derive a contradiction. It suffices to prove $\neg Q^* \wedge \neg R^*$. To prove $\neg Q^*$, assume Q^*. Then P^* follows, which is a contradiction of the assumption $\neg P^*$ *true*. Similarly, to prove $\neg R^*$, assume R^*. Then P^*, and hence a contradiction. So $\neg Q^* \wedge \neg R^*$ *true*, which is a contradiction.

□