15-399 Supplementary Notes: Hypothetical and General Judgements

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1 Review: Hypothetical Judgements

Constructive logic is founded on two forms of *categorical judgement*

1. $A\,\mathit{prop}$, asserting that A expresses a proposition;

2. $A\,\mathit{true}$, asserting that A is a true proposition (there is a proof of it).

Of these the first is *analytic*, or *self-evident*, whereas the second is *synthetic*, which means that it requires evidence in the form of a proof.

The *hypothetical judgement* expresses logical consequence: $A_1\,\mathit{true}, \ldots, A_n\,\mathit{true} \vdash A\,\mathit{true}$ asserts the existence of a proof of the proposition A, under the assumptions that the propositions $A_1,\ldots,A_n$ have proofs. Under this interpretation, the hypothetical judgement satisfies the following *structural rules*:

1. Reflexivity:
$$\frac{}{\Gamma, A\,\mathit{true} \vdash A\,\mathit{true}}$$

2. Transitivity:
$$\frac{\Gamma \vdash A\,\mathit{true} \quad \Gamma, A\,\mathit{true} \vdash B\,\mathit{true}}{\Gamma \vdash B\,\mathit{true}}$$

3. Weakening:
$$\frac{\Gamma \vdash A\,\mathit{true}}{\Gamma, B\,\mathit{true} \vdash A\,\mathit{true}}$$

4. Contraction:
$$\frac{\Gamma, A\,\mathit{true}, A\,\mathit{true} \vdash B\,\mathit{true}}{\Gamma, A\,\mathit{true} \vdash B\,\mathit{true}}$$

5. Permutation:
$$\frac{\Gamma, A\,\mathit{true}, B\,\mathit{true}, \Gamma' \vdash C\,\mathit{true}}{\Gamma, B\,\mathit{true}, A\,\mathit{true}, \Gamma' \vdash C\,\mathit{true}}$$

The synthetic judgement $A\,true$ can be put into analytic form by making the proof explicit. The judgement $M : A$ asserts that M is a proof of the proposition A. The meaning of the judgement $A\,true$ may then be restated as asserting $M : A$ for an unspecified proof term M. Correspondingly, the hypothetical judgement takes the form $u_1 : A_1, \ldots, u_n : A_n \vdash M : A$, and asserts that M is a proof of A possibly involving the free variables $u_1, \ldots, u_n$ representing the unknown proofs of $A_1, \ldots, A_n$.

The structural rules may be re-stated using proof terms as follows:

1. Reflexivity:
$$\frac{}{\Gamma, u : A \vdash u : A}$$

2. Transitivity:
$$\frac{\Gamma \vdash M : A \quad \Gamma, u : A \vdash N : B}{\Gamma \vdash [M/u]N : B}$$

3. Weakening:
$$\frac{\Gamma \vdash M : A}{\Gamma, u : B \vdash M : A}$$

4. Contraction:
$$\frac{\Gamma, u : A, v : A \vdash M : B}{\Gamma, u : A \vdash [u/v]M : B}$$

5. Permutation:
$$\frac{\Gamma, v : A, u : B, \Gamma' \vdash M : C}{\Gamma, v : B, u : A, \Gamma' \vdash M : C}$$

2 General Judgements

Mathematical objects, such as numbers or ordered pairs or functions, may be classified as *elements* of various *types*. For example, 17 is an element of type **nat**, and $\langle 4, 5\rangle$ is an element of type $\mathbf{nat} \times \mathbf{nat}$. This classification is stated in terms of these two forms of categorical judgement:

1. τ *type*, asserting that τ is a type.

2. $t \in \tau$, asserting that t is an element of type τ.

Both of these judgements are analytic, or self-evident.

A *predicate* is a proposition expressing a property of an *element* of a *type*. Equivalently, a predicate is a *propositional function* assigning a proposition to each element of a specified type (its domain). For example, the proposition $t =_N u$ expresses the equality of the two elements $t \in \mathbf{nat}$ and $u \in \mathbf{nat}$. That is, equality is a proposition about elements of the type $\mathbf{nat}\times\mathbf{nat}$ of ordered pairs of natural numbers. Thus equality is a propositional function whose domain is the type $\mathbf{nat} \times \mathbf{nat}$.

The *general judgement* $x_1 \in \tau_1, \ldots, x_n \in \tau_n \vdash J$, where J is any categorical judgement, assertions that $[t_1, \ldots, t_n/x_1, \ldots, x_n]J$ holds for any terms $t_1 \in \tau_1$, $\ldots$, $t_n \in \tau_n$.

The general judgement $x_1 \in \tau_1, \ldots, x_n \in \tau_n \vdash A\,prop$ states that A is a propositional function (predicate) of the variables $x_1 \in \tau_1$, $\ldots$, $x_n \in \tau_n$. For example, the general judgement $x \in \mathbf{nat}, y \in \mathbf{nat} \vdash x =_N y\,prop$ asserts that equality is a propositional function of type variables, x and y, of type **nat**.

The general judgement $x_1 \in \tau_1, \ldots, x_n \in \tau_n \vdash t \in \tau$ states that t is a *variable element* of type τ over the variables $x_1 \in \tau_1$, $\ldots$, $x_n \in \tau_n$. This means that $[t_1, \ldots, t_n/x_1, \ldots, x_n]t \in \tau$ whenever $t_i \in \tau_i$ (for each $1 \leq i \leq n$).

The general judgement $x_1 \in \tau_1, \ldots, x_n \in \tau_n \vdash A\,jtrue$ states that $[t_1, \ldots, t_n/x_1, \ldots .x_n]A\,true$ for any $t_i \in \tau_i$ $(1 \leq i \leq n)$. Similarly, $x_1 \in \tau_1, \ldots, x_n \in \tau_n \vdash M : A$ states that $[t_1, \ldots, t_n/x_1, \ldots, x_n]M : [t_1, \ldots, t_n/x_1, \ldots, x_n]A$.

The general judgement satisfies the following structural rules:[1]

1. Reflexivity:
$$\frac{}{\Delta, x \in \tau \vdash x \in \tau}$$

2. Transitivity:
$$\frac{\Delta \vdash t \in \tau \quad \Delta, x \in \tau \vdash J}{\Delta \vdash [t/x]J}$$

3. Weakening:
$$\frac{\Delta \vdash J}{\Delta, x \in \tau \vdash J}$$

4. Contraction:
$$\frac{\Delta, x \in \tau, y \in \tau \vdash J}{\Delta, x \in \tau \vdash [x/y]J}$$

5. Permutation:
$$\frac{\Delta, y \in \tau, x \in \sigma, \Delta' \vdash J}{\Delta, x \in \sigma, y \in \tau, \Delta' \vdash J}$$

Note that reflexivity is limited to the type membership judgement; the others are stated for an arbitrary categorical judgement J.

There is an obvious similarity between the hypothetical and general judgements, the only difference being that in the general judgement the variables range over types, whereas in the hypothetical judgement the variables range over proofs of propositions. Both forms satisfy the same structural rules. Indeed, we may combine the two judgement forms into one by permitting both proof variables and term variables in the context.

[1]We write Δ for any sequence of the form $x_1 \in \tau_1, \ldots, x_n \in \tau_n$, where no two x's are the same.

3 Definitional Equality

The notion of definitional equality for proofs extends to elements of a type in the obvious way. The basic simplification principles state that the elimination is a post-inverse of the introduction rule. These extend to a notion of reduction, $t \Rightarrow u$, and conversion, $t \Leftrightarrow u$, between terms just as proof simplifications induced reduction and conversion relations on proof terms.

In the presence of predicates reduction and conversion of terms induce corresponding notions of reduction and conversion for propositions and proofs. For example, the proposition $\mathit{prime}(3+4)$ asserts that the sum of 3 and 4 is prime; this is a true proposition because $3+4$ is definitionally equivalent to 7, and, of course, 7 has no divisors other than itself and 1. More precisely, the proposition $\mathit{prime}(3+4)$ is definitionally equivalent to the proposition $\mathit{prime}(7)$. Moreover, because the latter is true, so also is the former.

This may be captured by the following principles:

$$\frac{\Gamma \vdash A\ \mathit{true} \quad A \Leftrightarrow B}{\Gamma \vdash B\ \mathit{true}} \qquad \frac{\Gamma \vdash M : A \quad A \Leftrightarrow B}{\Gamma \vdash M : B}$$

These two rules state that *truth is invariant under definitional equality.*