15-399 Supplementary Notes: The Gödel-Genzen Interpretation of Classical Logic

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1 Proof of the Gödel-Gentzen Embedding

The Gödel-Gentzen embedding P^{G} of classical into constructive logic is defined by induction on the structure of propositions as follows:[1]

$$\begin{array}{rcl}
P^{\mathsf{G}} & = & \neg\neg P \quad (P \ \mathit{atomic}) \\
\bot^{\mathsf{G}} & = & \bot \\
\top^{\mathsf{G}} & = & \top \\
(P \wedge Q)^{\mathsf{G}} & = & P^{\mathsf{G}} \wedge Q^{\mathsf{G}} \\
(P \vee Q)^{\mathsf{G}} & = & \neg(\neg P^{\mathsf{G}} \wedge \neg Q^{\mathsf{G}}) \\
(P \supset Q)^{\mathsf{G}} & = & P^{\mathsf{G}} \supset Q^{\mathsf{G}}
\end{array}$$

This translation expresses the constructive content of classical logic. In classical logic we always have the option of proving a proposition by contradiction (proving $\neg\neg P$, but stating it as a proof of P). Classical logic is also weaker when it comes to disjunction: rather than prove one or the other disjunct, we may instead prove that both cannot fail to hold true. From a constructivist viewpoint the classical proof proves less than it claims. The Gödel-Gentzen translation makes this precise.

From a classical point of view the translation does nothing.

Theorem 1.1 *Classically, P and P^{G} are equivalent.*

Proof: Proceed by induction on the structure of P, using truth tables to show the equivalence. □

From a constructive viewpoint, it "constructivizes" classical logic. We write $\vdash_{class} P \ \mathit{true}$ to mean that $P \ \mathit{true}$ is derivable using the rules of constructive logic, plus the law of the excluded middle ($P \vee \neg P \ \mathit{true}$ for every P).

Theorem 1.2 *If $\vdash_{class} P \ \mathit{true}$, then $\vdash P^{\mathsf{G}} \ \mathit{true}$.*

[1] The course notes write P^N instead of P^{G}.

This means that the constructivist may always reinterpret what the classicist says in constructive terms.

Our aim is to give a proof of this theorem. The proof is a bit tricky in spots, which is why we outline it here.

A proposition P is *stable* iff $\neg\neg P \supset P$ *true*. Recall that it is easy to show constructively that $P \supset \neg\neg P$ *true*.

Lemma 1.1 *Every negated proposition is stable. That is, if $P = \neg Q$, then $\neg\neg P \supset P$ true.*

Proof: Assume $\neg\neg P$ *true*, that is $\neg\neg\neg Q$ *true*. We are to show P *true*, that is $\neg Q$ *true*. Assume towards a contradiction that Q *true*. It follows that $\neg\neg Q$ *true*. But this contradicts the assumption $\neg\neg\neg Q$ *true*. □

An implication is stable if its consequent is stable.

Lemma 1.2 *If R is stable, then $Q \supset R$ is stable.*

Proof: Assume $\neg\neg(Q \supset R)$ *true*. To show $Q \supset R$ *true*, assume Q *true*. Since R is stable, it is enough to show $\neg\neg R$ *true*. So assume towards a contradiction that $\neg R$ *true*. We will show $\neg(Q \supset R)$ *true* to obtain a contradiction. So assume $Q \supset R$ *true*. Then since Q *true*, we have R *true*, which contradicts the assumption that $\neg R$ *true*. □

Lemma 1.3 *For all propositions P, P^{G} is stable.*

Proof: By induction on the structure of P.

1. If P is atomic, then $P^{\mathsf{G}} = \neg\neg P$, so it is negated. By the Lemma 1.1, it is stable.
2. If P is $\bot$ or $\top$, it is easy to show stability.
3. If $P = Q \wedge R$, the result follows by induction, using the fact that $(P \wedge Q)^{\mathsf{G}} = P^{\mathsf{G}} \wedge Q^{\mathsf{G}}$.
4. If $P = Q \vee R$, then P^{G} is negated, and hence is stable by Lemma 1.1.
5. If $P = Q \supset R$, then by induction R is stable, and hence P is stable by Lemma 1.2.

□

If $\Gamma = P_1$ *true*, $\ldots, P_n$ *true*, then $\Gamma^{\mathsf{G}} = P_1^{\mathsf{G}}$ *true*, $\ldots, P_n^{\mathsf{G}}$ *true*.

Theorem 1.3 *If $\Gamma \vdash_{class} P$ true, then $\Gamma^{\mathsf{G}} \vdash P^{\mathsf{G}}$ true.*

Proof: By induction on the derivation of the assumption.

1. Suppose that $P = Q \vee \neg Q$ and that we have derived P *true* by applying the law of the excluded middle. We are to show that $\neg(\neg Q^{\mathsf{G}} \wedge \neg\neg Q^{\mathsf{G}})$ *true*. So assume $\neg Q^{\mathsf{G}} \wedge \neg\neg Q^{\mathsf{G}}$ *true*. But this is a contradiction!

2. Suppose that $P = Q \supset R$, and that we derived P *true* by implication introduction, assuming Q *true* and deriving R *true*. Then by induction we have proved constructively R^{G} *true* from the assumption Q^{G} *true*, and hence $Q^{\mathsf{G}} \supset R^{\mathsf{G}}$ *true*. That is, $(Q \supset R)^{\mathsf{G}}$ *true*.

3. Suppose that we have derived P *true* from $Q \supset P$ *true* and Q *true*. By induction $Q^{\mathsf{G}} \supset P^{\mathsf{G}}$ *true* and Q^{G} *true*, so P^{G} *true*, as required.

4. Suppose that $P = Q \vee R$ and that we have derived P *true* by $\vee$-introduction (left) from Q *true*. By induction Q^{G} *true*, and hence P^{G} *true*. The symmetric case is handled similarly.

5. Suppose that P *true* is derived from $Q \vee R$ *true*, P *true* assuming Q *true*, and P *true* assuming R *true*, using $\vee$-elimination. By induction $\neg(\neg Q^{\mathsf{G}} \wedge \neg R^{\mathsf{G}})$ *true* is derivable constructively. Moreover, P^{G} *true* is derivable constructively from Q^{G} *true* and also from R^{G} *true*. We are to show P^{G} *true*. By Lemma 1.3 it is enough to show $\neg\neg P^{\mathsf{G}}$ *true*. So assume $\neg P^{\mathsf{G}}$ *true*, and derive a contradiction. It suffices to prove $\neg Q^{\mathsf{G}} \wedge \neg R^{\mathsf{G}}$. To prove $\neg Q^{\mathsf{G}}$, assume Q^{G}. Then P^{G} follows, which is a contradiction of the assumption $\neg P^{\mathsf{G}}$ *true*. Similarly, to prove $\neg R^{\mathsf{G}}$, assume R^{G}. Then P^{G}, and hence a contradiction. So $\neg Q^{\mathsf{G}} \wedge \neg R^{\mathsf{G}}$ *true*, which is a contradiction.

□