15-399 Supplementary Notes: The "Law" of the Excluded Middle

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The Law of the Excluded Middle (LEM) is the proposition $A \vee \neg A$. From a classical point of view, LEM is a tautology — it is always true, regardless of whether A is true or false. Constructively, LEM is far from true ... but neither is it false! One important consequence is that we are free to assume (generally, or specific instances of) the LEM without fear of degenerating into inconsistency.

Neither Provable nor Refutable

From a constructive point of view, the judgement $A \vee \neg A$ *true* asserts that the problem expressed by the proposition A is *decidable* in that we either know a proof of A or we know a proof of $\neg A$ (*i.e.*, we can refute A). On this interpretation, it is obvious that not every proposition is decidable. Indeed, an *open problem* is precisely a proposition for which we have neither a proof nor a refutation. For example, there is at present no evidence that $P = NP$ *true*, nor is there evidence for $P \neq NP$ *true*. However, other classes of proposition are decidable. For example, equality and inequality of integers are both decidable: the judgements $a = b \vee a \neq b$ *true* and $a < b \vee a \not< b$ *true* are both evident over the integers.

Since there are (and always will be) open problems, we cannot expect LEM to be true, constructively speaking. Indeed, if we attempt to construct a (normal) proof of LEM, we get stuck rather quickly:

$$\dfrac{\dfrac{\dfrac{\textbf{???}}{A\ \mathit{true} \vdash \bot\ \mathit{true}}}{\neg A\ \mathit{true}}(\neg{-}I)}{A \vee \neg A\ \mathit{true}}(\vee{-}I{-}L)$$

Since some problems are, in fact, decidable, we cannot expect LEM to be refuted either. That is, we cannot expect $\neg(A \vee \neg A)$ *true* to be evident in general. In fact we can show something even stronger: constructively logic positively denies the falsehood (refutability) of (every instance of) LEM.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(A \lor \neg A)\ true}\,(R^u) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \neg(A \lor \neg A)\ true}\,(R^u) \quad \dfrac{\dfrac{}{\Gamma' \vdash A\ true}\,(R^v)}{\Gamma' \vdash A \lor \neg A\ true}\,(\lor\text{-}I\text{-}L)}{\Gamma' \vdash \bot\ true}\,(\neg\text{-}E)}{\Gamma \vdash \neg A\ true}\,(\neg\text{-}I^v)}{\Gamma \vdash A \lor \neg A\ true}\,(\lor\text{-}R)}{\Gamma \vdash \bot\ true}\,(\neg\text{-}E)}{\neg\neg(A \lor \neg A)\ true}\,(\neg\text{-}I^u)$$

where Γ is the context $\neg(A \lor \neg A)\ true^u$, and Γ' is the context $\Gamma, A\ true^v = \neg(A \lor \neg A)\ true^u, A\ true^v$.

Axiomatic Freedom

One consequence of these observations is that constructive logic affords a greater degree of expressive power than does classical logic. At first blush, it appears that the inability to prove LEM is a *loss* of expressive power compared to classical logic. But the irrefutability of LEM in constructive logic means that this apparent loss is, in fact, a *gain*! Proofs in classical logic may, and often do, rely implicitly on the universal validity of LEM — *every* proposition is "decidable" under the truth table interpretation — whereas in constructive logic we do without it.

But what if the only proof we know is one that relies on a case analysis on whether a proposition A is true or false? If A is not known to be decidable, then from a constructive viewpoint the theorem is conditioned on the *additional assumption* that $A \lor \neg A\ true$. Adding such an assumption is always "safe", precisely because constructive logic does not refute any instance of LEM! Thus constructive logic gives us the *freedom* to introduce uses of LEM in a controlled manner, whereas classical logic thrusts it on us whether we like it or not. In this sense constructive logic is *more expressive*, rather than *less expressive*, than classical logic.

Decidability and Undecidability

By the proof given earlier, for *no* proposition A do we have $\neg(A \lor \neg A)\ true$. Does this mean that no proposition is undecidable? Doesn't this contradict well-known results such as the undecidability of the halting problem?

The Halting Problem states that for an arbitrary Turing machine M, either M halts or M diverges. Classically this is a triviality, because it is essentially an instance of LEM. But constructively we can neither prove nor refute it, because there are some machines for which we can prove that they either halt or diverge, and there are some machines for which we have no proof either way. Thus, in logical terms we cannot, in a constructive setting, prove that for arbitrary M, either $M \downarrow$ or $M \uparrow$. This inability is precisely what is meant by the undecidability of the halting problem!

(To make this argument completely precise requires us to connect up proofs in constructive logic with programs of certain types, which we will do shortly.)