[3] See *Theory of Games and Economic Behaviour*, 2nd edition, Princeton, 1947, p. 21.
[4] See e.g. Paul Bernays, "Mathematics as a Domain of Theoretical Science and of Mental Experience" in *Logic Colloquium 1973*, ed. by H. E. Rose and J. C. Shepherdson, Amsterdam, 1975.

PER MARTIN-LÖF

ANALYTIC AND SYNTHETIC JUDGEMENTS IN TYPE THEORY

When Kant introduced his well-known distinction between analytic and synthetic judgements, he was well aware that it was not something entirely new. In the *Prolegomena*, he gives an explicit and very detailed reference to Locke, reproaching his dogmatic predecessors Wolff and Baumgarten for having neglected it, and one may take more or less for granted that he knew the Leibnizian distinction between truths of reason and truths of fact, although, strangely enough, he never gives, as far as I know, an explicit reference to it. Also, we know of the Humean distinction between relations of ideas and matters of fact, which is of course even verbally very close to the Leibnizian distinction. Kant's own terminology was that of analytic versus synthetic judgements. After Kant, we also find the distinction treated by Bolzano, for instance, who spoke about conceptual and intuitional propositions, Ger. *Begriffs- und Anschauungssätze*, respectively. With Bolzano, the situation is a bit strange in comparison with Kant, because he had not only the distinction between conceptual and intuitional propositions, but he also had the distinction between analytic and synthetic propositions. However, he interpreted the notions of analyticity and syntheticity in a completely different way, so that Bolzano's notion of an analytic proposition is what we now would call a logically valid proposition, that is, one which is true under all possible interpretations, which is a notion that is entirely different from the Kantian notion, whereas Bolzano's distinction between conceptual and intuitional propositions is very close to the Kantian distinction between analytic and synthetic judgements. Lastly, I would like to mention Mill, who spoke about, on the one hand, essential and accidental propositions: the difference between propositions and judgements is not significant here, and, on the other hand, verbal and real propositions. The difference he made between these two couples of notions was merely a terminological one. Now Mill's first terminology is particularly illuminating, because, if this distinction between analytic and synthetic judgements is so important, it would

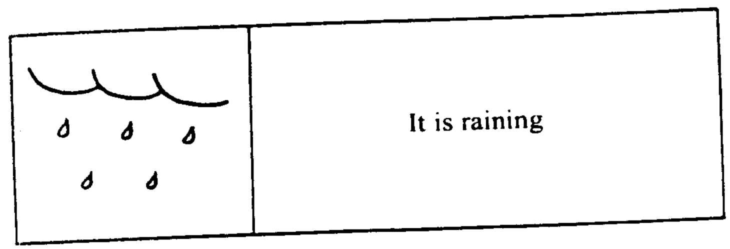
Paolo Parrini, ed., Kant and Contemporary Epistemology, 87–99.

be strange if there had been nothing corresponding to it in logic before the modern time. And, indeed, Mill's first terminology indicates very clearly, I think, what it corresponds to, namely, the distinction between essential and accidental properties that played such an important role in Aristotelian and scholastic logic. Now a judgement in which an essential property is ascribed to something is an analytical judgement, whereas a judgement in which an accidental property is ascribed to something, that is, a judgement which says that an accidental property inheres in something, is a synthetic judgement in Kant's terminology. So Mill's terminology, essential versus accidental propositions, is very aptly chosen: it hints directly at the heart of the matter. And, by the way, if we proceed beyond Mill, the distinction between essential and accidental properties reappears, and was apparently rediscovered by Wittgenstein, in the *Tractatus* under the name of the distinction between formal properties and properties proper, Ger. *formale und eigentliche Eigenschaften*, but it is the old distinction between essential and accidental properties that is at stake.

To explain the distinction between analytic and synthetic judgements, Kant makes a very clarifying move by introducing the two terms explicative judgement, Ger. *Erläuterungsurteil*, and ampliative judgement, Ger. *Erweiterungsurteil*, respectively. The idea is that the analytical, or explicative, judgements are those that become evident merely by conceptual analysis, that is, they are those whose evidence rests on conceptual analysis alone. That explanation, or almost verbatim that explanation, was given by Kant, and I certainly cannot improve the now current formulation that an analytic judgement is one which is evident in virtue of the meanings of the terms that occur in it, that is, the canonical formulation that we all seem to use. So that was as far as the analytic, or explicative, judgements are concerned. Our present understanding of them is fortunately in essential agreement with Kant's own understanding of them: what we seem to have difficulties with is understanding properly Kant's notion of a synthetic, or ampliative, judgement. The Kantian idea is that there are certain judgements which are such that they are not evident solely in virtue of the meanings of the terms involved, but, on the contrary, you have to go beyond what is contained entirely within the judgement in order to make it evident to yourself. If it is an empirical judgement, in Kant's terminology, an a posteriori judgement, then what you go to beyond the judgement itself is to experience: you

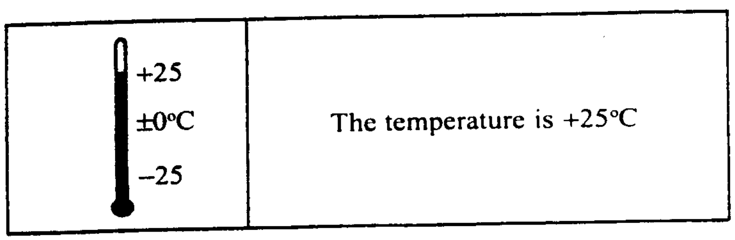
have to look out so to speak, whereas, in the case of a purely mathematical judgement, what needs to be joined to the judgement itself in order to make it evident is a construction, a mathematical construction. Kant had this wonderful formulation, which was quoted also by Prof. Körner, mathematical knowledge through the construction of concepts, Ger. *mathematische Erkenntnis durch die Konstruktion der Begriffe*, and which is the key to understanding the Kantian notion of a synthetic a priori judgement. So a synthetic judgement is one which is such that you have to go beyond the judgement itself in order to convince yourself of it, and, in the purely mathematical case, that going beyond means that you have to make a more or less ingenious construction in order for the judgement to become evident.

Before going into type theory, perhaps I could give some childishly simple nonmathematical examples of the distinction between analytic and synthetic judgements, and you will then see later that there is a great similarity between these nonmathematical examples and the corresponding type theoretical treatment. So look at this, for instance,



If I say, It is raining, then, of course, supposing now that it is raining, which it is not, no amount of mere conceptual analysis of what is contained in this judgement can tell you that it is raining: if you want to convince yourself of the fact that it is raining, there is no other way than to expose yourself directly to the falling rain, or else to provide some kind of indirect evidence. So it is the falling rain, or the piece of indirect evidence, that makes the judgement that it is raining evident, which means that this judgement is synthetic. On the other hand, if you look at the whole complex as a judgement, consisting of a nonlinguistic part, the falling rain, and a properly linguistic part, the utterance, then that judgement is an analytic one:

you see, everything is contained in that judgement that you need in order to convince yourself of it. If someone does not agree to the statement that it is raining in the presence of the falling rain, then there is simply something wrong with his conceptual understanding: he lacks some of the concepts involved or he has misunderstood them or something like that. It must at once be admitted, however, that this way of treating the linguistic and the nonlinguistic on a par is something which would have been alien to Kant, so I have definitely said something that goes beyond Kant now. An entirely similar analysis applies if you take the example of someone's saying, The sun is shining, or if you take,



The judgement, The temperature is +25°C, is by itself certainly a synthetic one: it is not enough just to analyze the concepts which are contained in it in order to convince oneself of it, but, if that verbal expression is taken together with the thermometer showing +25°C, then that whole complex is again clearly something analytic, and it is easy enough of course to multiply this list. You may take the example, say, of this pencil: if I say, This pencil is 15 cm long, and you just see the pencil, then of course it is not enough just to recall the meanings of the terms involved in order to make that judgement evident to yourself, but, if I have the ruler and squeeze it to the ruler in the appropriate way, and you can read the ruler, then that taken together with the purely verbal part, This pencil is 15 cm long, that whole complex is certainly something analytical again. Already from these simple examples, you see something very important, namely, that every synthetic judgement is grounded on an analytic judgement, and that the synthetic judgement is obtained by so to speak suppressing a certain part of the analytical judgement, or the analytical connection.

What has all this got to do with type theory, which figures in the title of my talk? Well, type theory as we have it now differs a bit, and I will explain exactly how, from the simple theory of types, but it is most easily described by displaying directly the forms of judgement that it employs. First of all, it makes judgements of the form

$$a : \alpha.$$

Such a judgement says that a is an object of type α, where Kant would have said category instead of type, category or pure concept of understanding. In addition to judgements of this first form, you have also identity judgements, that is, judgements of the form

$$a = b : \alpha.$$

A judgement of this latter form says that a and b are identical objects of type α, where identity is to be understood as definitional identity. These are the two basic forms of categorical judgement employed in type theory.

Now how are the types of the system generated? Well, the type structure is a generalization of the type structure of the simple theory of types. If you remember Church's formulation of the simple theory of types, he has a ground type o for the type of propositions and another ground type ι for the type of individuals, and then he builds the type structure over them by the clause that, if α and β are types, then the type of functions from α to β, which Church denoted by $(\beta\alpha)$ but for which I shall prefer Schütte's notation $(\alpha)\beta$, is again a type. So the simple type structure is generated by the three rules

$$o : \text{type}, \quad \iota : \text{type}, \quad \frac{\alpha : \text{type} \quad \beta : \text{type}}{(\alpha)\beta : \text{type}}.$$

In intuitionistic type theory, this is generalized: we still have the type of propositions, written prop in type theory, although it is the same as Church's o, but, because of the so-called Curry–Howard correspondence between propositions and sets, it can be identified with the type of sets. So, in intuitionistic type theory, there is a ground type of sets, governed by the axiom

$$\text{set} : \text{type},$$

with which the type of propositions, and hence also Church's o, is identified,

$$\text{prop} = \text{set} : \text{type},$$

but, instead of having a single other ground type ι of individuals, we have a rule,

$$\frac{A : \text{set}}{\text{elem}(A) : \text{type}},$$

which says that, for each set A, the elements of A form a new ground type. In the logical reading, when A is thought of as a proposition, elem(A) serves also as the type of proofs of A,

$$\text{proof}(A) = \text{elem}(A) : \text{type}.$$

If you want to stay as close as possible to Church's notation for the simple types, then you would write elem(A) simply as ι(A), which shows that we do not have just a single ground type ι of individuals: rather, for each set A, we have a type ι(A) of individuals belonging to that set. Now these were the ground types, obtained by generalizing Church's first two rules of type formation, and then we also have the generalization of the rule for forming function types,

$$\frac{\alpha : \text{type} \quad \begin{matrix}(x : \alpha)\\ \beta : \text{type}\end{matrix}}{(x : \alpha)\beta : \text{type}},$$

which says that, if α and β are types, of which β may depend on a variable x ranging over α, that is, on a variable x of type α, then the dependent function type $(x : \alpha)\beta$ will again be a type, namely, the type of functions whose argument is of type α and whose value for the argument x is of type β, which may depend on x. The reason why this generalizes Church's third rule of type formation is that, in the special case when β does not depend on x, we get Church's function type as a special case of the dependent function type: $(\alpha)\beta$ can simply be defined as $(x : \alpha)\beta$ in the case when β does not depend on x,

$$(\alpha)\beta = (x : \alpha)\beta : \text{type}.$$

So these are the rules that generate the types α that figure to the right in the two basic forms of judgement $a : \alpha$ and $a = b : \alpha$.

If you want, you may consider what I have just been through as a modern analogue of Kant's metaphysical deduction of the categories, or pure concepts of understanding. I mean, Kant had this fortunate idea, which he called the clue to the discovery of all pure concepts of understanding, of looking at the forms of judgement of logic as it was current at his time, and, by making a complete list of the forms of judgement that were used, he arrived at his categories. In our case, the α that appears in $a : \alpha$ and $a = b : \alpha$ is precisely a form, or, rather, two coupled forms of judgement, so we simply have to sit down and reflect and see what forms of judgement that we are using in logic at present, and it turns out that these forms can all be generated from the two ground forms, which say that something is a set, respectively, an element of a set, with the corresponding logical readings, by the rule for forming function types. So this is an exhaustive list of the categories that we are using in type theory at present, and you see that Kant's identification of a category, or pure concept of understanding, with a form of judgement remains entirely intact.

Now, whatever type α that we choose, the two forms of judgement $a : \alpha$ and $a = b : \alpha$ are both analytic. You see, if a judgement of one of these two forms is evident at all, then it is evident solely by virtue of the meanings of the terms that occur in it. So you may wonder, where do the synthetic judgements come from? Well, they arise in the following way. Consider again the first form of judgement, $a : \alpha$, which expresses that a is an object of type α. Then we often do not care about exactly what this object a of type α is, but are merely interested in the existence of an object of type α. So we may introduce a new form of judgement, say,

$$\alpha \text{ exists},$$

to be read alternatively, either the concept α has existence, or there exists an α, or α is, simply, as Bolzano and, following him, Brentano said: these are all different readings of one and the same existential form of judgement. The meaning of an existential judgement, that is, a judgement of this new existential form, can be gleaned from the rule

$$\frac{a : \alpha}{\alpha \text{ exists}}.$$

More precisely, to explain semantically a form of judgement, in general, you have to lay down what it is to know a judgement of that form, and, in this case, the explanation is that to know that α exists is to know an object which falls under α. The existential form of judgement includes the usual form of judgement,

$$A \text{ is true},$$

where A is a proposition, as a special case. To see that, remember that we had among our ground types, for an arbitrary set or proposition A, the type elem(A) or, in the logical reading, proof (A). That is a ground type, so we may form the judgement that that type, or concept, has existence,

$$\text{proof}(A) \text{ exists}.$$

Now this existential judgement says that there exists a proof of the proposition A, and hence, according to the intuitionistic explanation of the notion of truth, it says exactly that the proposition A is true, because truth is defined intuitionistically as existence of a proof, or construction, of the proposition. So the usual form of judgement, A is true, is indeed a special case of the existential form of judgement.

How is this related to the distinction between analytic and syn-
thetic judgements? I already said that the two basic forms of judge-
ment, which say that a is an object α of type and that a and b are
identical objects of type α, respectively, that those two forms of judge-
ment are both analytic. The synthetic form of judgement is precisely
the existential form of judgement that I have just introduced. So an
existential judgement is synthetic in Kant's terminology. To see why,
we have to ask ourselves, what does its evidence rest on? Is it merely
on conceptual analysis, or do we have to go beyond what is con-
tained entirely within the judgement in order to make it evident?
Well, its evidence rests on a construction: you see, we arrive at an
existential judgement, α exists, say, through the construction of an
object which falls under the concept α, that is, through the construc-
tion of the concept, Ger. *durch die Konstruktion des Begriffs*, in Kantian
terminology. So we clearly have to go beyond what is contained in
the judgement itself, namely, to the thing that exists, in order to make
an existential judgement evident, and hence it must be synthetic. That
Kant himself considered existential judgements to be synthetic is
actually explicitly stated in the *Critique of Pure Reason*. If you look
in the section on the impossibility of an ontological proof of the ex-
istence of God, you will find him saying quite explicitly that every
existential proposition is synthetic, Ger. *daß ein jeder Existenzialsatz synthetisch sei*. So every existential proposition, or existential judgement, was according to Kant synthetic, and clearly so, because it corresponds exactly to his explanation of the source of the synthetic judgements. You can also see the analogy clearly here with the nonmathematical examples that I started with: again, the synthetic judgement is obtained by suppressing a certain part of the underlying analytical judgement. So we have again confirmed that every synthetic judgement is grounded on an analytic judgement, and in that sense the notion of analytic judgement is of course the more basic or the more important notion.

The preceding analysis of the notions of analyticity and syntheticity has an important consequence, which I hope you will be somewhat startled by, and that is the consequence that the logical laws in their usual formulation are all synthetic. I do not want to go into the distinction between the a priori and the a posteriori here, but, if we just take for granted Kant's use of these terms, then it is clear that they are also a priori. So the logical laws in their usual formulation are not only synthetic but synthetic a priori. And, as a matter of fact, I have already given the explanation why: it is because the logical laws in their usual formulation all say that an arbitrary proposition of a certain form is true, and the affirmative form of judgement, A is true, is a form of synthetic judgement, but let me make it even more clear, I hope, by considering a particular example. So let us take one simple example of a logical law, say, $A \supset (B \supset A \,\&\, B)$. That is a standard law of propositional and therefore also of predicate logic. Now the law of course really states that any proposition of this form is true, so, by the intuitionistic analysis of the notion of truth, what it says is that a proof of this proposition exists, and, as you see, this is a judgement of existential form and therefore synthetic. And how do you convince yourself of the truth of this proposition in the standard way? Well, you have to make a certain construction, and, if you choose to work in the notation of natural deduction, it would read as follows,

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{A} \quad \overset{(2)}{B}}{A \,\&\, B}\ \text{by \& I}}{B \supset A \,\&\, B}\ \text{by} \supset \text{I, discharging (2)}}{A \supset (B \supset A \,\&\, B)}\ \text{by} \supset \text{I, discharging (1)}$$

So, in order to convince yourself of the truth of the proposition that we are considering, you have to make this construction, and I have here used a notation which is familiar to all of us. In the notation of type theory, the task is to find a construction c which is a proof of the proposition $A \supset (B \supset A \mathbin{\&} B)$, symbolically, c : proof($A \supset (B \supset A \mathbin{\&} B)$), and the solution is

$$\supset \mathrm{I}(A, B \supset A\&B, (x) \supset \mathrm{I}(B, A \mathbin{\&} B, (y) \mathbin{\&} \mathrm{I}(A, B, x, y))) \\ : \mathrm{proof}(A \supset (B \supset A \mathbin{\&} B)),$$

but, as you see, this is nothing but the above natural deduction style proof written in the linear, or functional, notation characteristic of type theory. The essential point is, I hope, clear here, namely, that, whatever logical law you take in its usual formulation, there is a little construction, requiring ingenuity, however slight, that has to be made in order for the law to become evident.

It should be added, though, that propositional and, more generally, predicate logic does not only consist of the logical laws, which say that some schematic proposition is true, but it contains of course also the rules of term formation, which allow you to prove something to be an individual, or, in the formalistic reading, an individual term, and the rules of formula formation, which generate for you the propositions, or formulas. So predicate logic really consists of rules for deriving judgements of the three forms

$t : \iota =$ ind,
$A : o =$ prop,
A is true,

and the first two of these are special cases of the first form of judgement that I introduced, the form $a : \alpha$, which means that they are both analytic: it is only the third which is a form of synthetic judgement. As Prof. Körner said, and I agree with him, Kant did not consider all mathematical judgements to be synthetic, but the interesting ones, so to speak, are all synthetic, and that is entirely right. The interesting mathematical judgements are precisely the existential ones, and the difficulty of a mathematical proof is to find, or construct, an object which falls under the concept to which the existential judgement ascribes existence. It is precisely in this that the computer proof systems that we are in the process of developing at present are supposed to be of help: they are supposed to provide some mechanical assistance in the arduous task of constructing an object which falls

under a given concept, or is of a given type.

Do I have some more time? (Ten minutes.) Oh, ten minutes. Let me then say that the distinction between analytic and synthetic judgements turns out to be crucial for a proper understanding of the incompleteness and undecidability phenomena that we know of for the traditional logical calculi. You see, the situation is this: the logic of analytic judgements, that is, the logic for deriving judgements of the two analytical forms, is complete and decidable, whereas the logic of synthetic judgements is incomplete and undecidable, as was shown by Gödel. That the incompleteness theorem pertains to the logic of synthetic judgements is clear, because the crucial judgement, upon the consideration of which the proof of the incompleteness theorem is based, is of the form A is true, which is a special case of the existential form of judgement and therefore synthetic. Now what do I mean by saying that the logic of analytic judgements is complete and decidable? Simply that, if you have a judgement of one of the two analytical forms, that is, one of the two forms $a : \alpha$ and $a = b : \alpha$, then it can be checked, or decided, whether or not that judgement is derivable by means of the formal rules, and the algorithm for doing that is what the computer scientists call the type checking algorithm, which is so to speak the formal, or logical, heart of the computer proof systems that I just mentioned. So that is what I mean by saying that the logic of analytic judgements is decidable, and it is also complete in the sense that, if you have a judgement of one of the two analytical forms which contains certain constants, expressing concepts, then no other laws are needed to derive it than the laws which concern precisely those concepts, and, since you need not go beyond the laws that concern the concepts in terms of which the judgement is expressed, those laws are actually complete for deriving the judgement, provided now that it is of one of the two analytical forms. What is it then that gives rise to the incompleteness and undecidability phenomena? Well, it is of course the fact that, in the synthetic, or existential, form of judgement, α exists, the object a of type α that is claimed to exist has been suppressed, and, in the passage from the analytic $a : \alpha$ to the synthetic α exists, the decidability is lost: to prove that α exists, we have to search for an object a of type α, and, even if we limit ourselves to a particular formal system, that search procedure will not necessarily terminate, let alone then if we put no limitation on the axioms that we are allowed to invoke in the course of that procedure. In particular, choosing the type to be of the form proof(A), where A is a proposition, it is the fact that the proof object a : proof(A) has been suppressed in the form of judgement proof(A) exists, or, what amounts to the same, A is true, that is the source of the undecidability, and it is the same with the incompleteness. You see, consider again the simplest and most important case of a judgement of the form A is true, where A is a proposition, and suppose that the proposition A has been formulated in a certain language, or system. Then A may be unprovable in that system, simply because there is no proof or construction a : proof(A) that can be expressed in it. On the other hand, it may of course well be that, if we go from the original system to some extension of it by introducing new concepts, like going from arithmetic to arithmetical analysis or the theory of generalized inductive definitions or something like that, then we shall be able to construct an object a : proof(A) in the extended system, although that was impossible in the original system in which the proposition A was expressed. It was this phenomenon that Gödel discovered by taking the original system to be first order arithmetic, say, and constructing a particular first order arithmetical proposition A which is such that there is no proof of A in first order arithmetic: on the other hand, he showed informally to begin with that there exists nevertheless such a proof, which is to say that A is true, and it can also be formalized in a suitable extension of first order arithmetic, obtained by adding a reflection principle, for instance. So this is what I had in mind when saying that the distinction between analytic and synthetic judgements turns out to be crucial for a proper understanding of the incompleteness and undecidability phenomena.

Since this is a meeting on Kant and Contemporary Epistemology, let me just end with some homage to Kant. By choosing to concentrate, as I have done, on certain Kantian themes in logic and philosophy of mathematics, I certainly do not want to imply that what Kant achieved in these areas is in any way the most important of his philosophical contributions: surely, his general philosophical view point, which is to say, his transcendental idealism, is of much greater significance, but at least I wanted to show you that, even in this limited area of logic and philosophy of mathematics, he has had a very important insight, namely, in the existence of synthetic a priori judgements, and that they arise because interesting mathematical theorems require for their proof a construction to be carried out. So you have

this formulation, which I have already quoted twice, mathematical knowledge through the construction of concepts, Ger. *mathematische Erkenntnis durch die Konstruktion der Begriffe*, a splendid formulation, which no doubt had a fruitful influence on Brouwer, and to my mind it is justifiable to say that intuitionism is a development of an essentially Kantian position in the foundations of mathematics.

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