15-399 Supplementary Notes: Constructive Negation

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1 Introduction

The most notable difference between constructive and classical logic is the treatment of negation. From a constructive viewpoint we have $\neg P$ *true* true if assuming P *true* leads to a contradiction. In other words we must positively refute P *true* in order to show that $\neg P$ *true*. Since P might express an unsolved problem in mathematics, we cannot, in general, expect to either prove or refute every proposition. Moreover, the impossibility of refuting P is distinguished from affirming P, because these proofs have different information content. These considerations lead to some surprising properties of constructive negation.

Recall that negation is defined by the equation

$$\neg P = P \supset \bot.$$

This leads to the following derived formation, introduction, and elimination rules

$$\frac{P\ \mathit{prop}}{\neg P\ \mathit{prop}}\ \neg F \qquad \frac{\begin{array}{c}\overline{P\ \mathit{true}}^{\,u} \\ \vdots \\ \bot\ \mathit{true}\end{array}}{\neg P\ \mathit{true}}\ \neg I^{u} \qquad \frac{P\ \mathit{true} \quad \neg P\ \mathit{true}}{Q\ \mathit{true}}\ \neg E$$

2 The "Law" of the Excluded Middle

In classical logic it is a trivial fact that $P \vee \neg P$ *true*, since it expresses that every proposition denotes either "true" or "false". This is called the *law of the excluded middle (LEM)*. Since constructive logic identifies truth with the existence of a proof, the situation is rather different. Let us say that a proposition is *decidable* iff $P \vee \neg P$ *true* in the constructive sense. This means that we either have a proof of P *true* or a proof of $\neg P$ *true*. Clearly not every proposition is decidable — for example, we do not at present have a proof that $P = NP$ nor do we have a proof that $P \not= NP$ — and so the “law” of the excluded middle cannot be expected to hold universally in the constructive setting.

It is instructive to attempt to construct a proof of $P \vee \neg P$ *true* according to the rules of constructive logic. Convince yourself that the attempt is a complete non-starter, for absent any other information we have no choice but to attempt a direct proof. This means that we must either prove P *true* or prove $\neg P$ *true*. But we can expect to do neither, since P is an arbitrary proposition! (Later we will be in a position to prove that there is in fact no proof of LEM.)

Now the failure of LEM to hold for *arbitrary* P does not mean that it cannot hold for *some* P, nor does it mean that constructive logic *refutes* it, *i.e.*, proves $\neg(P \vee \neg P)$ *true*! Certainly some propositions are decidable. For example, the proposition $a < b$ is decidable when a and b range over the integers. We need only look at a and b to determine which is larger (say, by subtracting a from b and noting the sign of the result). Hence we cannot expect that LEM is refuted either.

It is instructive to attempt to prove $\neg(P \vee \neg P)$ *true*. The only option is to assume $P \vee \neg P)$ *true*, and derive a contradiction. From there the only option is to perform a case analysis on $P \vee \neg P$ *true*, deriving a contradiction from the assumption P *true* and from the assumption that $\neg P$ *true*. But since P is arbitrary, we cannot expect to complete the proof, for doing so would refute all propositions.

So constructive logic neither affirms nor refutes LEM. This means, in particular, that it is consistent to *assume* that LEM holds, just as it does in classical logic. From a constructive point of view this is precisely the content of classical mathematics — it is mathematics done under the assumption that all propositions are decidable.

Interestingly, it *does* refute the denial of LEM, which is to say that $\neg\neg(P \vee \neg P)$ *true* is provable! This means that we *cannot* consistently add to constructive logic the denial of LEM. Here is an outline of a proof of the double negation of LEM.

1. To prove $\neg\neg(P \vee \neg P)$ *true*, we assume $\neg(P \vee \neg P)$ *true* and derive a contradiction. Label this assumption u. We will use this assumption *twice* in the proof.

2. The only way to use the assumption u is to prove $P \vee \neg P$ *true* making use of assumption u to derive $\perp$ *true*. Note that we *must* make use of the assumption u, because without it we cannot prove $P \vee \neg P$ *true*.

3. To prove $P \vee \neg P$ *true* we prove $\neg P$ *true*, making use of the assumption u.

4. To prove $\neg P$ *true*, we assume P *true* and derive $\perp$ *true*. Label this assumption v.

5. By assumption v, using rule $\vee I_L$, we have $P \vee \neg P$ *true*. Applying rule $\neg E$ to this proof and assumption u, we derive $\perp$ *true*.

6. Therefore by $\neg I^v$ we have $\neg P$ *true*, and hence by rule $\vee I_R$, we have $P \vee \neg P$ *true*.

7. Applying rule $\neg E$ to this proof and assumption u (again), we obtain $\perp$ *true*.

8. Apply rule $\supset I^u$ to obtain the result.

Summarizing, we have the following situation:

1. LEM is not provable for general propositions P.

2. LEM is not refutable for any proposition P, because the denial of LEM is provable.

3. Therefore we may *assume* LEM without fear of contradiction.

4. But we may *not* assume the denial of LEM without contradiction.

As we shall see later in the course classical logic may be considered to be a special case of constructive logic in which we assume that all propositions are decidable. The foregoing remarks tell us that this assumption is consistent, and hence cannot lead to paradox.

3 Double Negation Elimination

It is easy to show constructively that $P \supset \neg\neg P$ *true*. Simply assume P *true* and $\neg P$ *true*, and derive a contradiction immediately by applying rule $\neg E$. The converse, however, cannot be proved: there is no proof (in general) of $\neg\neg P \supset P$ *true*. To see why not, observe that we *can* prove $\neg\neg(P \vee \neg P)$ *true*, as described in the preceding section, but we *cannot* prove $P \vee \neg P$ *true*. Thus, in general, $\neg\neg P$ is a *strictly weaker* proposition than P itself.

As we shall see later in the course it is possible to interpret classical logic inside of constructive logic by "doubly negating" all statements, rendering them provable in constructive logic. From this point of view the theorems of classical logic are strictly weaker than those of constructive logic (from a constructive point of view!).