

Anytime PSRO

| | C_1 | C_2 | C_3 | C_4 | \dots | C_n |
|----------|-------|-------|-------|-------|---------|-------|
| R_1 | -1 | | | | | |
| R_2 | | | | | | |
| R_3 | | | | | | |
| R_4 | | | | | | |
| \vdots | | | | | | |
| R_m | | | | | | |

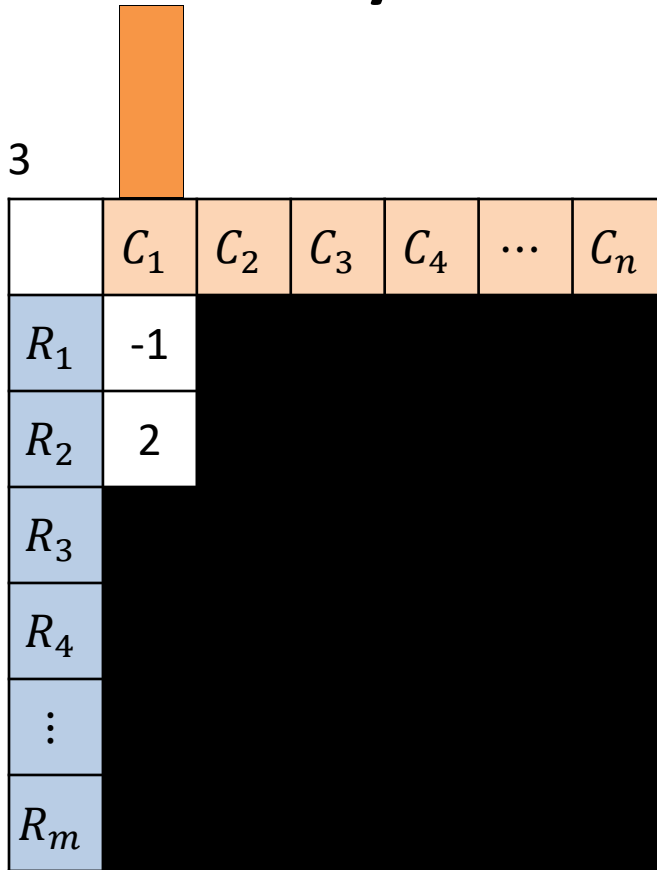
Anytime PSRO

Nash gap: 3

| | C_1 | C_2 | C_3 | C_4 | \dots | C_n |
|----------|----------|-------|-------|-------|---------|-------|
| R_1 | -1 | 1 | -1 | 1 | \dots | 1 |
| R_2 | 2 | | | | | |
| R_3 | 1 | | | | | |
| R_4 | -1 | | | | | |
| \vdots | \vdots | | | | | |
| R_m | -1 | | | | | |

Anytime PSRO

Nash gap: 3



| | C_1 | C_2 | C_3 | C_4 | \dots | C_n |
|----------|-------|-------|-------|-------|---------|-------|
| R_1 | -1 | | | | | |
| R_2 | 2 | | | | | |
| R_3 | | | | | | |
| R_4 | | | | | | |
| \vdots | | | | | | |
| R_m | | | | | | |

Anytime PSRO

Nash gap: 3

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|-------|-------|-------|-------|-------|-----|-------|
| R_1 | -1 | 1 | -1 | 1 | ... | 1 |
| R_2 | 2 | -2 | -1 | 1 | ... | 1 |
| R_3 | | | | | | |
| R_4 | | | | | | |
| ⋮ | | | | | | |
| R_m | | | | | | |

Idea: Solve the one-sided restricted game to compute meta-strategies

Something's wrong...

Requirement: Always find a novel best response if possible

Diversity is good! e.g.:

$$\pi_i^{t+1} = \arg \max_{\pi_i} \left\{ u(\pi_i, \sigma_{-i}^t) + \lambda \min_{\pi_i^k \in \mathcal{H}(\Pi_i^t)} \text{dist}(\pi_i, \pi_i^k) \right\}$$

Anytime PSRO

Nash gap: 3

Novel BR

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|----------|-------|-------|-------|-----|-------|
| R_1 | -1 | 1 | -1 | 1 | ... | 1 |
| R_2 | 2 | | | | | |
| R_3 | 1 | | | | | |
| R_4 | -1 | | | | | |
| \vdots | \vdots | | | | | |
| R_m | -1 | | | | | |

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Anytime PSRO

Nash gap: 3

Novel BR

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|-------|-------|-------|-------|-----|-------|
| R_1 | -1 | | -1 | | | |
| R_2 | 2 | | -1 | | | |
| R_3 | | | | | | |
| R_4 | | | | | | |
| \vdots | | | | | | |
| R_m | | | | | | |

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Anytime PSRO

Nash gap: 3

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|----------|-------|----------|-------|-----|-------|
| R_1 | -1 | | -1 | | | |
| R_2 | 2 | | -1 | | | |
| R_3 | 1 | | 0 | | | |
| R_4 | -1 | | -1 | | | |
| \vdots | \vdots | | \vdots | | | |
| R_m | -1 | | -1 | | | |

Idea: Solve the one-sided restricted game to compute meta-strategies

Something's wrong...

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Diversity is good! e.g.:

$$\pi_i^{t+1} = \arg \max_{\pi_i} \left\{ u(\pi_i, \sigma_{-i}^t) + \lambda \min_{\pi_i^k \in \mathcal{H}(\Pi_i^t)} \text{dist}(\pi_i, \pi_i^k) \right\}$$

Anytime PSRO

Nash gap: 1

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|-------|-------|----------|-------|-----|-------|
| R_1 | -1 | 1 | -1 | 1 | ... | 1 |
| R_2 | 2 | | -1 | | | |
| R_3 | | | 0 | | | |
| R_4 | | | -1 | | | |
| \vdots | | | \vdots | | | |
| R_m | | | -1 | | | |

Idea: Solve the one-sided restricted game to compute meta-strategies

Something's wrong...

Requirement: Always find a novel best response if possible

Diversity is good! e.g.:

$$\pi_i^{t+1} = \arg \max_{\pi_i} \left\{ u(\pi_i, \sigma_{-i}^t) + \lambda \min_{\pi_i^k \in \mathcal{H}(\Pi_i^t)} \text{dist}(\pi_i, \pi_i^k) \right\}$$

Anytime PSRO

Nash gap: 1

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|-------|-------|-------|-------|-----|-------|
| R_1 | -1 | | -1 | | | |
| R_2 | 2 | | -1 | | | |
| R_3 | 1 | | 0 | | | |
| R_4 | | | | | | |
| \vdots | | | | | | |
| R_m | | | | | | |

Idea: Solve the one-sided restricted game to compute meta-strategies

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$$\pi_i^{t+1} = \arg \max_{\pi_i} \left\{ u(\pi_i, \sigma_{-i}^t) + \lambda \min_{\pi_i^k \in \mathcal{H}(\Pi_i^t)} \text{dist}(\pi_i, \pi_i^k) \right\}$$

Anytime PSRO

Nash gap: 1

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|-------|-------|-------|-------|-----|-------|
| R_1 | -1 | 1 | -1 | 1 | ... | 1 |
| R_2 | 2 | -2 | -1 | 1 | ... | 1 |
| R_3 | 1 | 1 | 0 | 1 | ... | 1 |
| R_4 | | | | | | |
| \vdots | | | | | | |
| R_m | | | | | | |

Idea: Solve the one-sided restricted game to compute meta-strategies

Something's wrong...

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$$\pi_i^{t+1} = \arg \max_{\pi_i} \left\{ u(\pi_i, \sigma_{-i}^t) + \lambda \min_{\pi_i^k \in \mathcal{H}(\Pi_i^t)} \text{dist}(\pi_i, \pi_i^k) \right\}$$

Anytime PSRO

Nash gap: 0

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|----------|-------|-------|----------|-------|-----|-------|
| R_1 | -1 | | -1 | | | |
| R_2 | 2 | | -1 | | | |
| R_3 | 1 | | 0 | 1 | ... | 1 |
| R_4 | | | -1 | | | |
| \vdots | | | \vdots | | | |
| R_m | | | -1 | | | |

Exploitability is monotonically nonincreasing 😊

Every iteration requires us to solve a full game 😞

...in which P1 has not too many strategies. Can we solve it efficiently?

How do we solve games where one side has a small number of strategies?

Recall (HW1): If P1 runs a regret minimizer and P2 best-responds on every step, then

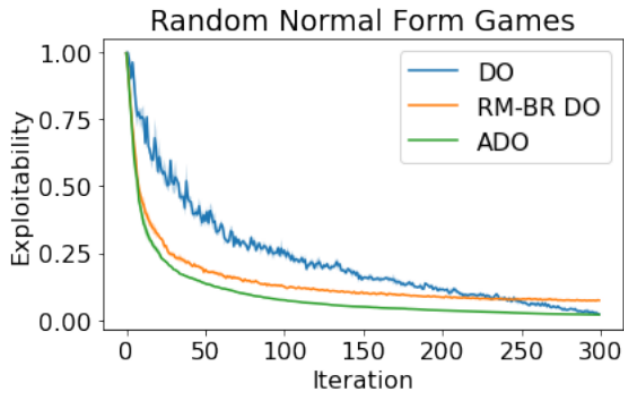
$$\text{Nash gap} \leq \text{P1's regret} / T$$

⇒ extremely efficient equilibrium computation when P1's strategy set is small!

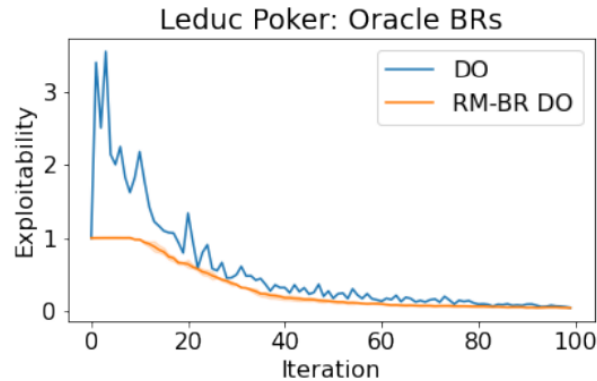
Anytime PSRO = one-sided PSRO

+ this idea (“regret minimization with best responses”/“RM-BR”)
+ RL best-response oracle for P2

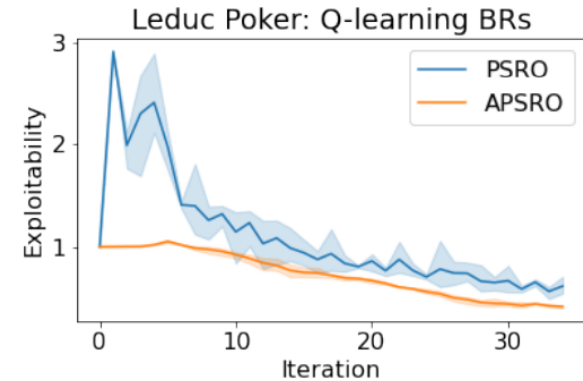
Anytime PSRO Experiments



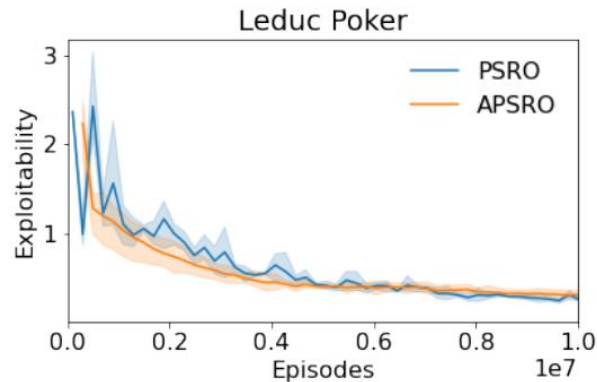
(a) Random Normal Form Games



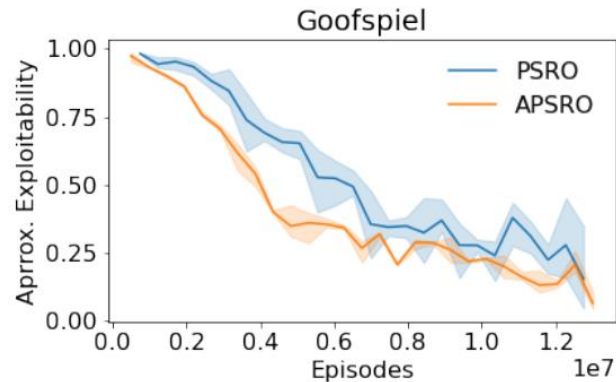
(b) Leduc with Oracle Best Responses



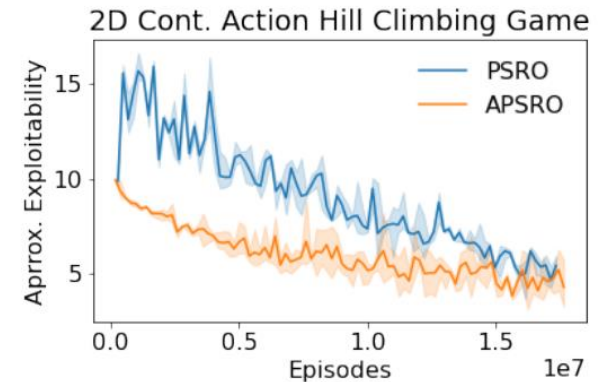
(c) Leduc with Q-Learning Best Responses



(a) Leduc with DDQN BRs



(b) Goofspiel with DDQN BRs



(c) Continuous-Action Hill-Climbing Game

RM-BR

$y^t = \text{BR}(x^t)$

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|-------|-------|-------|-------|-------|-----|-------|
| R_1 | | | | | | |
| R_2 | | | | | | |
| R_3 | | | | | | |

strategy x^t
selected by RM

From RM-BR to Self-Play PSRO

$y^t = \text{BR}(x^t)$

| | C_1 | C_2 | C_3 | C_4 | ... | C_n |
|------------------------|-------|-------|-------|-------|-----|-------|
| R_1 | | | | | | |
| R_2 | | | | | | |
| R_3 | | | | | | |
| $v^t = \text{BR}(y^t)$ | | | | | | |

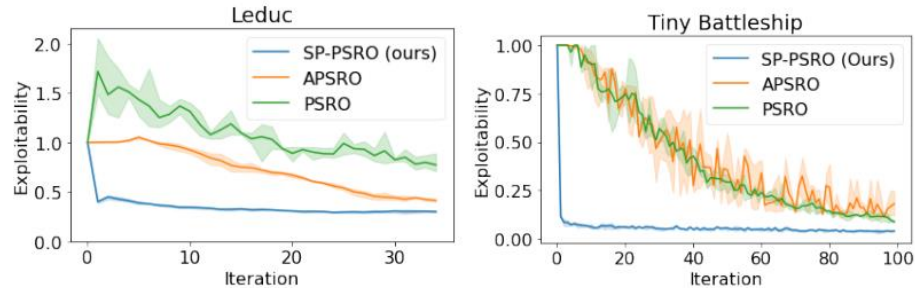
strategy x^t
selected by RM

After some time, add \bar{v}^t to P1's strategy set and y^t to P2's strategy set

“Self-play PSRO”

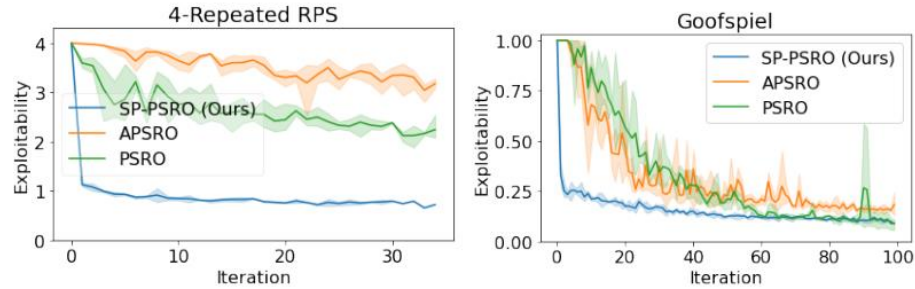
Intuition: self-play “stabilized” by having strategies R_1, R_2, R_3 available to the row player
 \Rightarrow better PSRO performance in practice?

Self-play PSRO experiments



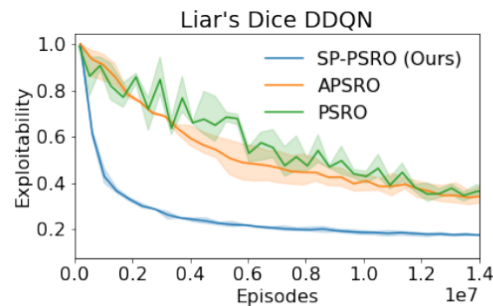
(a) Leduc Poker

(b) Battleship

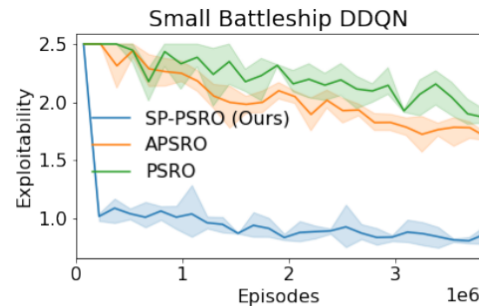


(c) Repeated RPS

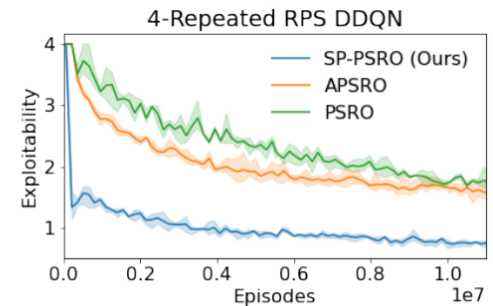
(d) Goofspiel



(a) DRL Liars Dice



(b) DRL Battleship



(c) DRL Repeated RPS

Certificates in Extensive-Form Games

Brian Hu Zhang and Tuomas Sandholm, “Small Nash Equilibrium Certificates in Very Large Games”, NeurIPS-20:

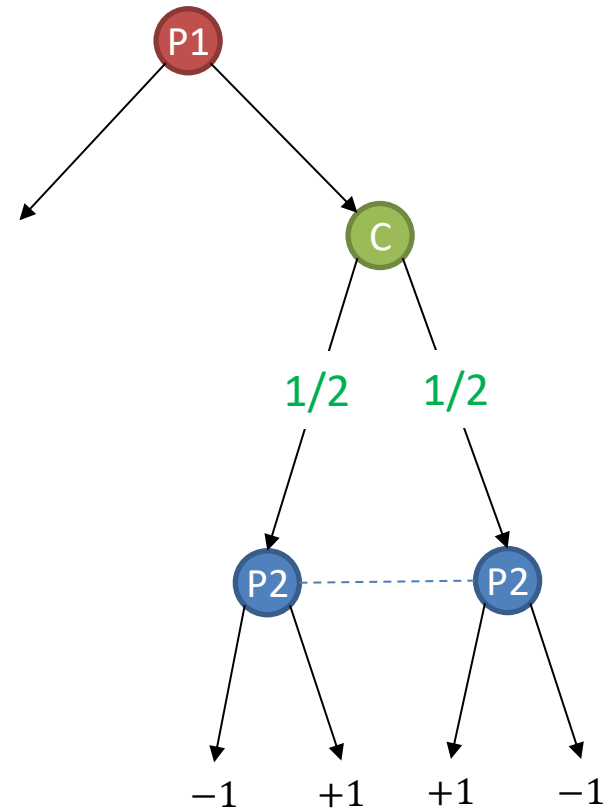
<https://arxiv.org/abs/2006.16387>

Brian Hu Zhang and Tuomas Sandholm, “Finding and Certifying (Near-)Optimal Strategies in Black-Box Extensive-Form Games”, AAI-21:

<https://arxiv.org/abs/2009.07384>

Pseudogames and Certificates

Pseudogame: Game without known utilities on all terminal nodes



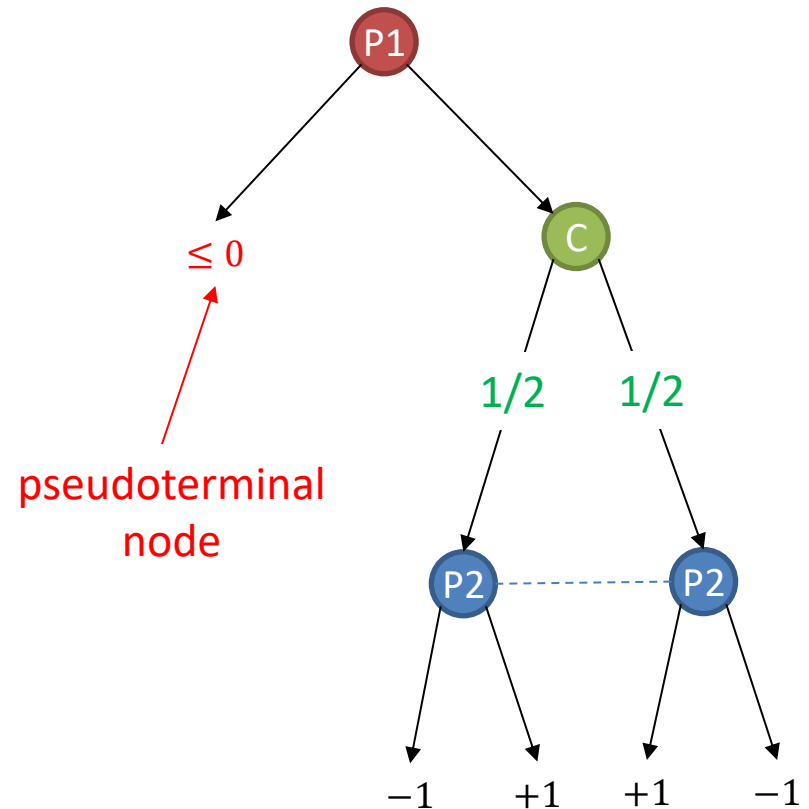
Pseudogames and Certificates

Pseudogame: Game without known utilities on all terminal nodes

Think: partially-expanded game tree, “alpha-beta” style

In zero-sum land, gives rise to **two** games:

- a *lower-bound game* in which rewards are optimistic for P2, and
- an *upper-bound game* in which rewards are optimistic for P1

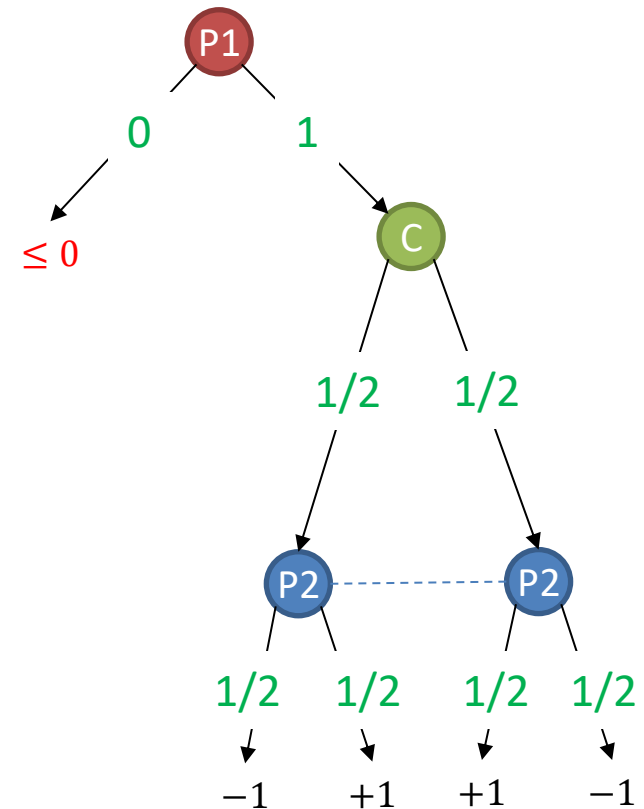


Pseudogames and Certificates

(Approximate) Nash equilibrium in a pseudogame: strategy profile in which every player is *provably* playing an (approximate) best response (irrespective of what happens at pseudoterminal nodes)

Results in Nash equilibrium regardless of what the pseudoterminal node hides!

(Approximate) Certificate:
Pseudogame created from partial expansion of a full game + (approximate) Nash equilibrium of that pseudogame



Small Certificates

- **Question:** When do small ε -certificates (size $O(N^c \text{poly}(1/\varepsilon))$ for some $c < 1$, where N is the number of nodes) exist?

When do Small Certificates Exist?

- **Answer #1:** They exist in **perfect-information zero-sum games with no nature randomness**,
...under reasonable assumptions about the game tree (uniform branching factor and depth, alternating moves)
 - **Proof:** The optimal alpha-beta search tree is a certificate of size $\approx \sqrt{N}$.

Small Certificates

Answer #2: They exist in (squarish) **normal-form games**.

Proof:

Consider an $m \times m$ normal-form game.

Lipton et al, 2003:

ε -Nash equilibrium exists where each player mixes between $\log(m) / \varepsilon^2$ pure strategies.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| T | | | | | | | |
| U | | | | | | | |
| V | | | | | | | |
| W | | | | | | | |
| X | | | | | | | |
| Y | | | | | | | |
| Z | | | | | | | |

Small Certificates

Answer #2: They exist in (squarish) **normal-form games**.

Proof:

Consider an $m \times m$ normal-form game.

Lipton et al, 2003:

ε -Nash equilibrium exists where each player mixes between $\log(m) / \varepsilon^2$ pure strategies.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| T | | | | | | | |
| U | | | | | | | |
| V | | | | | | | |
| W | | | | | | | |
| X | | | | | | | |
| Y | | | | | | | |
| Z | | | | | | | |

Small Certificates

Answer #2: They exist in (squarish) **normal-form games**.

Proof:

We only need those rows and columns!

$\Rightarrow O(m \log(m) / \varepsilon^2)$ -size certificate

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| T | | | | | | | |
| U | | | | | | | |
| V | | | | | | | |
| W | | | | | | | |
| X | | | | | | | |
| Y | | | | | | | |
| Z | | | | | | | |

Small Certificates

So, small certificates exist in games where the players have **perfect information** or **no information**.

What about in between?

Unfortunately, no. 😞

Bad News

Counterexample: Consider this game:

- Matching pennies
- repeated k times, each round worth $1/k$ points.
- After each round, both players learn what the other played

Game tree size: 4^k

Theorem: Any ε -certificate of this game must at least $4^{k(1-2\varepsilon)}$ nodes.

More Bad News

Theorem: It is NP-hard to approximate the smallest certificate of an extensive-form zero-sum game, to better than an $O(\log N)$ multiplicative factor.

Proof Idea: Reduction from set cover.

Oracle Model

Assume access to an **oracle** that allows us to query any node h to obtain:

- upper and lower bounds on the future utility after h
- the player to act at h , if any, and that player's information
- if the player to act is nature, the exact nature distribution

Goal:

- *Compute and verify “ex-post” approximate equilibria with only black-box access*
- Output both an equilibrium strategy **and** a bound ε on exploitability

More Bad News

Theorem: With only an oracle for an extensive-form zero-sum game, there is no equilibrium-finding algorithm that runs in time polynomial in the size of the smallest certificate.

Proof: One-player “SAT” games: certificate of size $O(\log N)$ exists, but clearly no sublinear-time algorithm.

Let's Try Anyway

Repeat until satisfied:

- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
- **Create** the next pseudogame, by expanding all pseudoterminal nodes in the support of the **optimistic profile** (in which the max-player her equilibrium strategy in the upper-bound game, and the min-player plays her strategy in the lower-bound game)

Output: Pessimistic profile, and ε = difference in values between upper- and lower-bound pseudogames

Intuition: In the perfect-information setting with no nature randomness, **it's just alpha-beta search**

Let's Try Anyway

Repeat until satisfied:

- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
- **Create** the next pseudogame, by expanding all pseudoterminal nodes in the support of the **optimistic profile** (in which the max-player her equilibrium strategy in the upper-bound game, and the min-player plays her strategy in the lower-bound game)

Output: Pessimistic profile, and $\varepsilon =$ difference in values between upper- and lower-bound pseudogames

Theorem (Correctness): If the pessimistic profile is not a Nash equilibrium, then the second step expands at least one node.

Let's Try Anyway

Repeat until satisfied:

- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
- **Create** the next pseudogame, by expanding all pseudoterminal nodes in the support of the **optimistic profile** (in which the max-player her equilibrium strategy in the upper-bound game, and the min-player plays her strategy in the lower-bound game)

Output: Pessimistic profile, and ε = difference in values between upper- and lower-bound pseudogames

“Works” even on games that have infinitely large trees or infinite/unbounded rewards!

Experiments

| game | size of game | | size of certificate | | | |
|-------------------------|--------------|-----------|---------------------|-------|----------|-------|
| | nodes | infosets | nodes | | infosets | |
| search game | 234,705 | 11,890 | 13,682 | 5.8% | 532 | 4.5% |
| 4-rank PI Goofspiel | 2,229 | 1,653 | 275 | 12.3% | 110 | 6.7% |
| 5-rank PI Goofspiel | 55,731 | 41,331 | 2,593 | 4.7% | 957 | 2.3% |
| 6-rank PI Goofspiel | 2,006,323 | 1,487,923 | 21,948 | 1.1% | 7,584 | 0.5% |
| 4-rank Goofspiel | 2,229 | 738 | 614 | 27.5% | 117 | 15.9% |
| 5-rank Goofspiel | 55,731 | 9,948 | 11,415 | 20.5% | 2,160 | 21.7% |
| 6-rank Goofspiel | 2,006,323 | 166,002 | 266,756 | 13.3% | 15,776 | 9.5% |
| 3-rank random Goofspiel | 1,066 | 426 | 309 | 29.0% | 92 | 21.6% |
| 4-rank random Goofspiel | 68,245 | 17,432 | 16,416 | 24.1% | 3,270 | 18.8% |
| 5-rank random Goofspiel | 8,530,656 | 1,175,330 | 1,854,858 | 21.7% | 241,985 | 20.6% |
| 5-rank Leduc | ∞ | ∞ | 26,306 | — | 2,406 | — |
| 9-rank Leduc | ∞ | ∞ | 137,662 | — | 6,811 | — |
| 13-rank Leduc | ∞ | ∞ | 337,312 | — | 12,171 | — |

Simulators

Assume access to a **simulator**:

- Allows us to play through the game **from the perspective of all players at once**
- Gives player to act, acting player's information, bounds on future utility, and valid actions
- **Does not** give nature distribution; only gives a single sample
- **Does not** allow saving and rewinding. Must perform complete play-throughs

Goal:

- *Compute and verify “ex-post” approximate equilibria with only black-box access*
- Output both an equilibrium strategy **and** a bound ε on exploitability
- **Want:** correctness with high probability, say, $1 - T^{-\gamma}$ for some $\gamma > 0$ after T iterations.

Lower Bounds

Theorem: Consider any algorithm with the following guarantee: For some constant $\gamma > 0$, given a zero-sum game in our black-box setting, with T game samples, the algorithm outputs a pair of strategies (x, y) **and a bound** ε_T such that, with probability $1 - O(T^{-\gamma})$, (x, y) is an ε_T -Nash equilibrium. Then

$$\varepsilon_T = \Omega\left(\sqrt{\frac{\log T}{T}}\right).$$

Our goal: Match this bound.

Main Tool: Pseudogames as Confidence Bounds

- At **nodes that have not yet been expanded**, use bounds given by simulator
- At **nature nodes h** , give each player reward bounded by $[-\rho, \rho]$, where

$$\rho = \Delta_h \sqrt{\frac{1}{2t_h} \log \frac{1}{\delta}}$$

range of utilities
possible from h

times h has
been reached

confidence parameter

Main Tool: Pseudogames as Confidence Bounds

- At **nodes that have not yet been expanded**, use bounds given by simulator.
- At **nature nodes h** , give each player reward bounded by $[-\rho, \rho]$, where

$$\rho = \Delta_h \sqrt{\frac{1}{2t_h} \log \frac{1}{\delta}}$$

Intuition: ρ represents the **uncertainty** in the nature distribution at h

Main Tool: Pseudogames as Confidence Bounds

- At **nodes that have not yet been expanded**, use bounds given by simulator.
- At **nature nodes** h , give each player reward bounded by $[-\rho, \rho]$, where

$$\rho = \Delta_h \sqrt{\frac{1}{2t_h} \log \frac{1}{\delta}}$$

Intuition: It looks like UCB. That is not a coincidence.

Choice of Confidence Bound

During equilibrium computation, values of children are changing, so we need to use a Hoeffding bound to be robust:

$$\rho = \Delta_h \sqrt{\frac{1}{2t_h} \log \frac{1}{\delta}}$$

During best response computation, strategy profiles after h are fixed by induction, so we can use a tighter empirical Bernstein bound [Maurer & Pontil '09]:

$$\rho = S \sqrt{\frac{2}{t_h} \log \frac{2}{\delta}} + \frac{7\Delta'_h}{3(t_h - 1)} \log \frac{2}{\delta}$$

where S is the unbiased sample standard deviation, and Δ'_h is the range of possible utilities from h **under the fixed strategy profile**, which may be much smaller than Δ_h

Main Tool: Pseudogames as Confidence Bounds

Theorem: For appropriate choice of $\delta = 1/\text{poly}(T, N)$, with high probability, at every time, for every strategy profile, for every player the true reward of the player is bounded by the pessimistic and optimistic rewards achieved in the confidence bound pseudogame.

(“Confidence bounds are actually bounds”)

Zero-Sum LP-Based Algorithm

Repeat T times:

- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
- **Sample** one play-through from the optimistic profile (in which the max-player her equilibrium strategy in the upper-bound game, and the min-player plays her strategy in the lower-bound game)
- **Create** the next pseudogame:
 - **Expand** all encountered nodes
 - **Update** empirical nature distributions of nature nodes sampled during play

Output: Pessimistic profile, and ε_T = difference in values between upper- and lower-bound pseudogames

Intuition: In the perfect-information setting with no nature randomness, **it's just alpha-beta search**

Zero-Sum LP-Based Algorithm

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- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
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- **Create** the next pseudogame:
 - **Expand** all encountered nodes
 - **Update** empirical nature distributions of nature nodes sampled during play

Output: Pessimistic profile, and $\varepsilon_T =$ difference in values between upper- and lower-bound pseudogames

Intuition: In the one-player “multi-armed bandit” setting, **it’s UCB (up to a constant factor).**

Zero-Sum LP-Based Algorithm

Repeat T times:

- **Solve** both the upper- and lower-bound pseudogames *exactly* (with e.g., an LP solver)
- **Sample** one play-through from the optimistic profile (in which the max-player her equilibrium strategy in the upper-bound game, and the min-player plays her strategy in the lower-bound game)
- **Create** the next pseudogame:
 - **Expand** all encountered nodes
 - **Update** empirical nature distributions of nature nodes sampled during play

Output: Pessimistic profile, and ε_T = difference in values between upper- and lower-bound pseudogames

Advantage: Sample-efficient

Disadvantage: Expensive iterations (requires game re-solve on each iteration)

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Theorem: The *best iterate* of the algorithm converges at rate

$$\mathbb{E}\varepsilon_T \leq \tilde{O}\left(\frac{N}{\sqrt{T}}\right)$$

number of nodes in final pseudogame
(may be \ll total number of nodes!)

Regret-Based Algorithm

Idea: Just use a regret minimizer, like CFR, for each player

Regret-Based Algorithm

Repeat T times:

- **Query** the regret minimizers for all players to obtain a strategy profile
- **Sample** one play-through from that strategy profile
- **Pass** the *optimistic* rewards to the regret minimizers
- **Create** the next pseudogame:
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Output: Average strategy profile

Several problems!

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Problem 1: The strategy space of each player is changing over time

Solution: CFR “handles it naturally”. *Formalization:* “Extendable” regret minimizers

Regret-Based Algorithm

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Problem 2: We don't want to run a full CFR iterate on every sample; that is expensive

Solution: Use MCCFR + outcome sampling. Still works.

Regret-Based Algorithm

Repeat T times:

- **Query** the regret minimizers for all players to obtain a strategy profile
- **Sample** one play-through from that strategy profile
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- **Create** the next pseudogame:
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Output: Average strategy profile

Problem 3: What equilibrium gap bound can we compute?

Regret-Based Algorithm

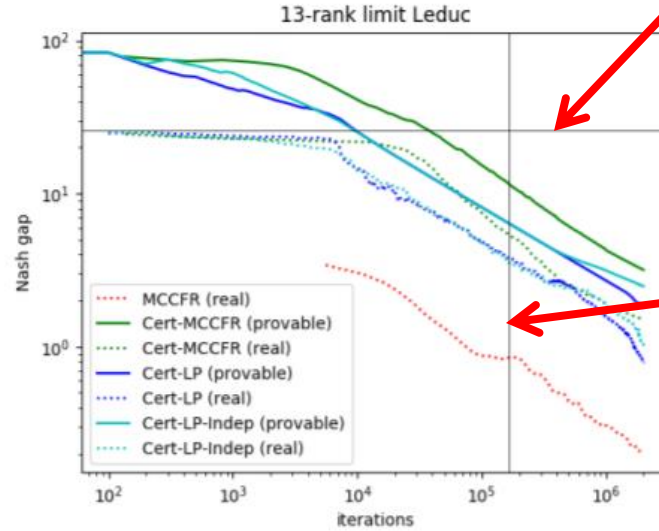
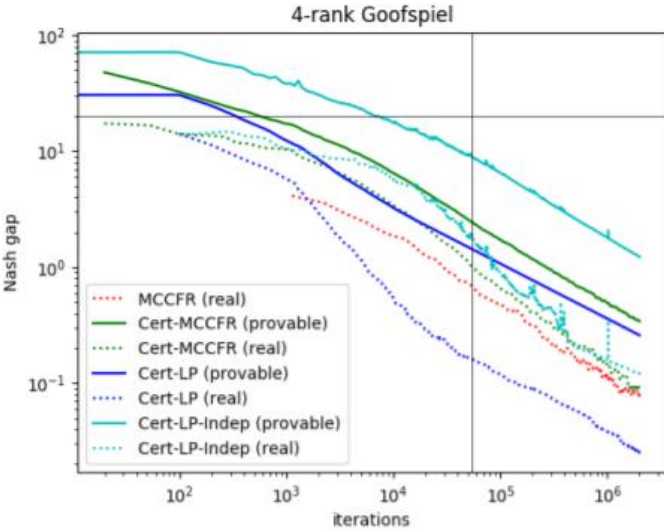
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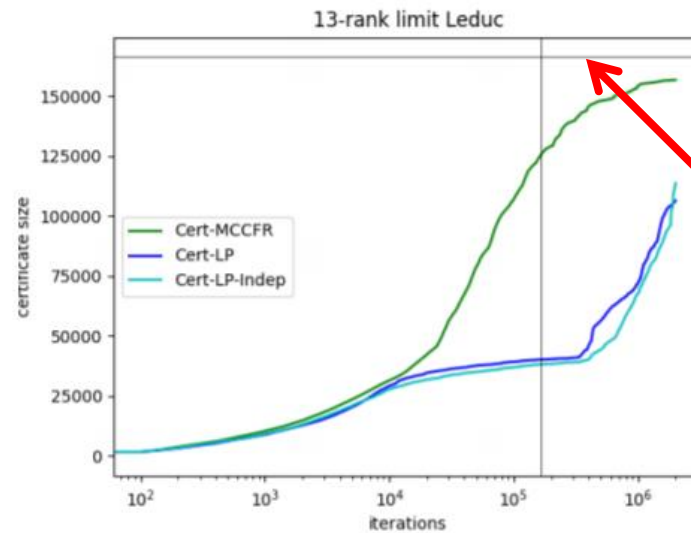
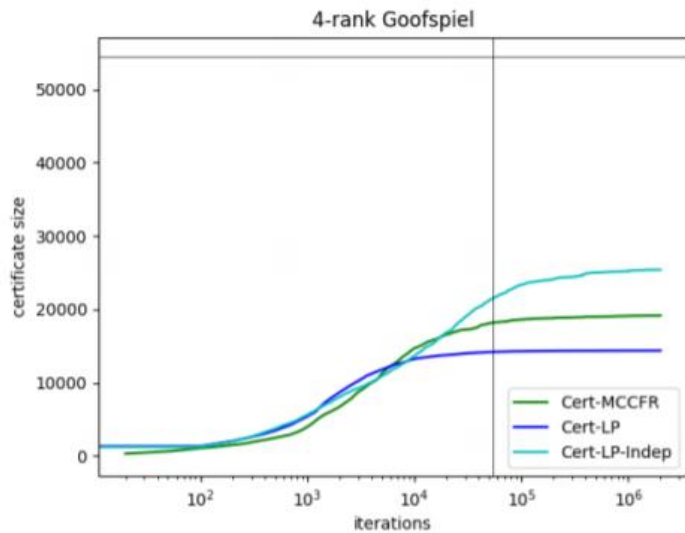
Theorem: The regret-based algorithm outputs a strategy profile (x, y) and exploitability bound ε after T iterations, where $\varepsilon \leq \tilde{O}(N^2/\sqrt{T})$.

Experiments



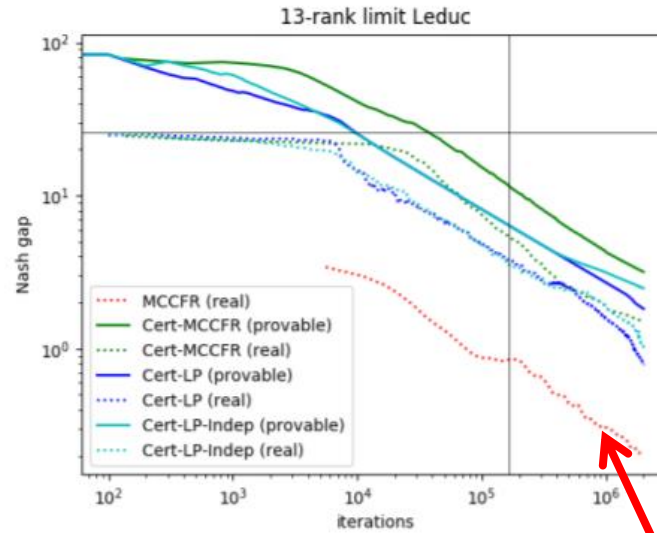
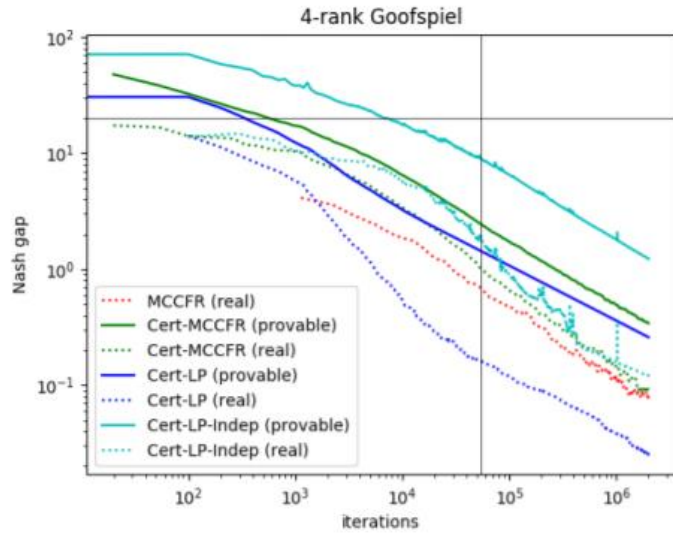
Horizontal line:
reward bound
of full game

Vertical line:
number of
nodes in full
game



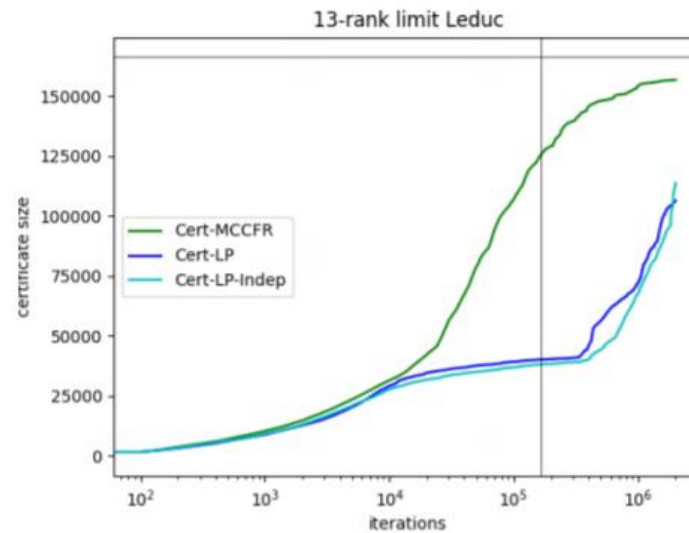
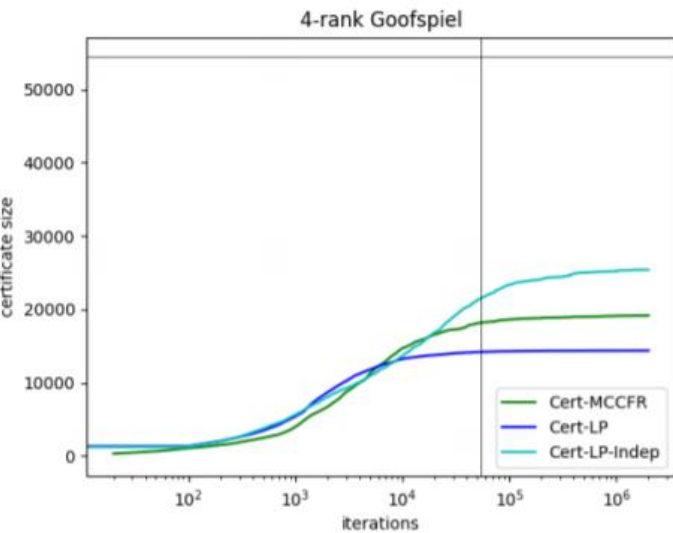
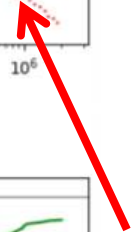
Horizontal line:
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Experiments

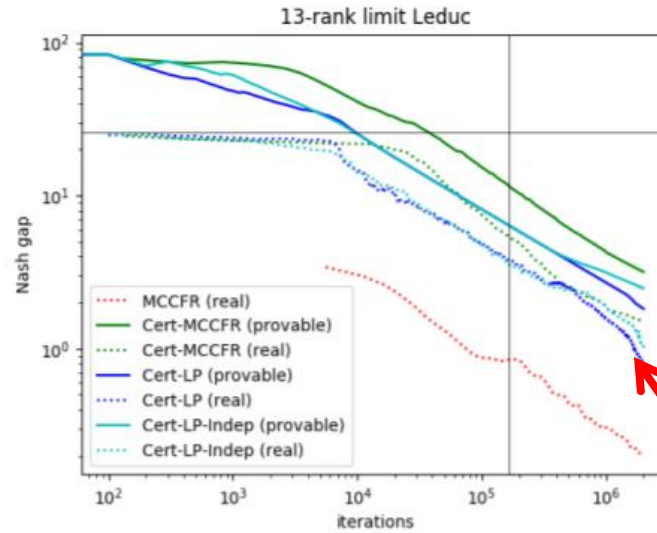
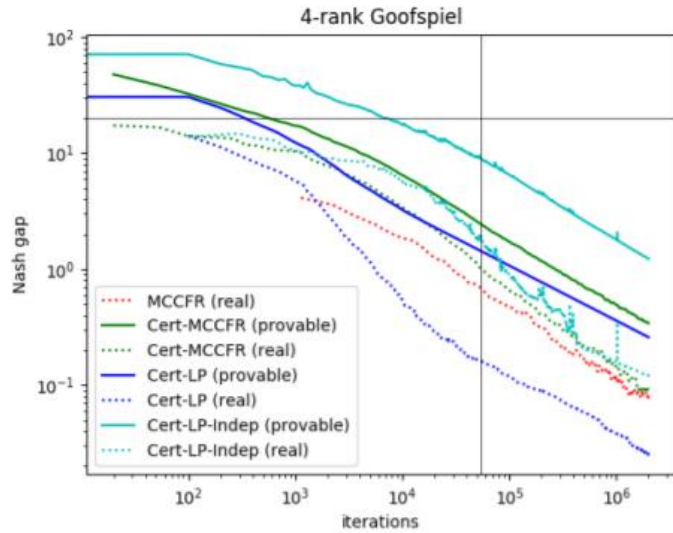


In all games, with all algorithms, nontrivial certificates are found **without expanding the full game tree**, in fact, with fewer game samples than there are game tree nodes

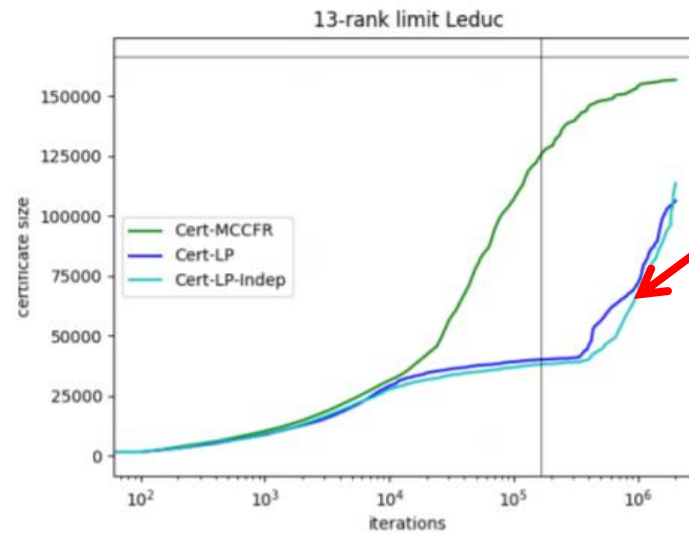
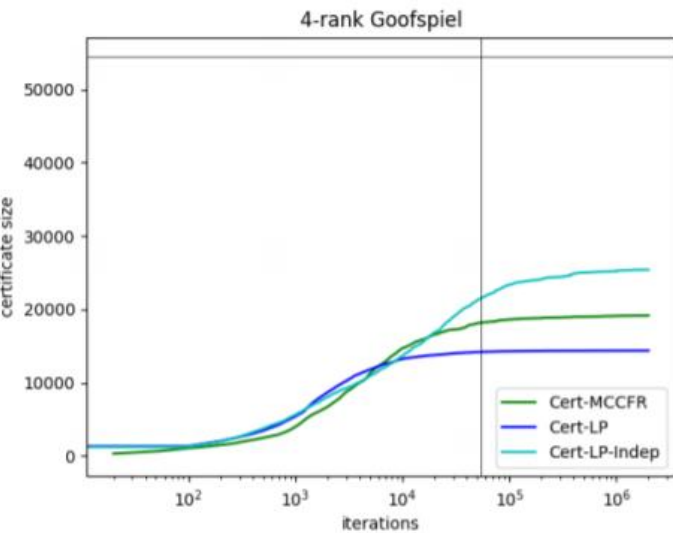
MCCFR converges quickly in reality, but this cannot be verified without expanding the rest of the game tree



Experiments



In all games, with all algorithms, nontrivial certificates are found **without expanding the full game tree**, in fact, with fewer game samples than there are game tree nodes



LP-based certificate finding has better sample efficiency and final certificate size than regret-based, but (not shown) runs slower

Summary, Comparison of Algorithms, and Further Research

| | Work per iteration | Convergence bound in terms of number of game samples |
|-------------------------|--------------------------------|--|
| Oracle LP algorithm | Solve a game with $O(N)$ nodes | n/a (requires oracle access) |
| Simulator LP algorithm | Solve a game with $O(N)$ nodes | $\tilde{O}(N/\sqrt{T})$ |
| Simulator CFR algorithm | $O(bd)$ | $\tilde{O}(N^2/\sqrt{T})$ |

N = number of nodes in *current pseudogame*
(could be less than the whole game size!)

Summary, Comparison of Algorithms, and Further Research

| Algorithm | $T^{-1/2}$ theoretical convergence rate? | Uncoupled dynamics? | Simulator access? |
|---|--|---|---|
| Outcome-sampling MCCFR | Yes | Yes | No (requires prior knowledge of game structure) |
| Farina & Sandholm [AAAI 2021] | No (only $T^{-1/4}$) | Yes | Yes |
| Regret-based Certificate Algorithm [this lecture] | Yes | No (exploration requires control of both players) | Yes |

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| IXOMD [Kozuno et al., <i>NeurIPS</i> 2021] | Yes | Yes | Yes |