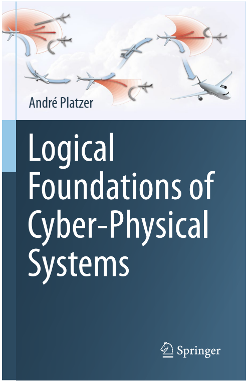
02: Differential Equations & Domains

Logical Foundations of Cyber-Physical Systems



Stefan Mitsch

Carnegie Mellon University
Computer Science Department

Outline

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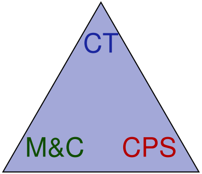
Learning Objectives
Differential Equations & Domains

semantics of differential equations
descriptive power of differential equations
syntax versus semantics



continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects

Outline

Example (Vector field and one solution of a differential equation)

$$\begin{pmatrix} y'(t) = f(t,y) \\ y(t_0) = y_0 \end{pmatrix}$$

Intuition:

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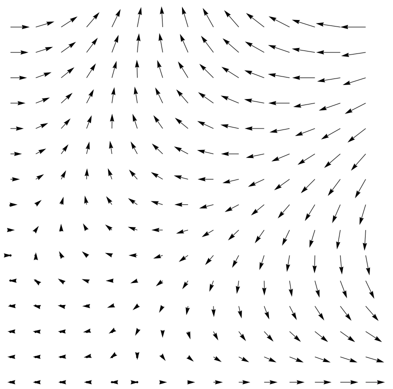

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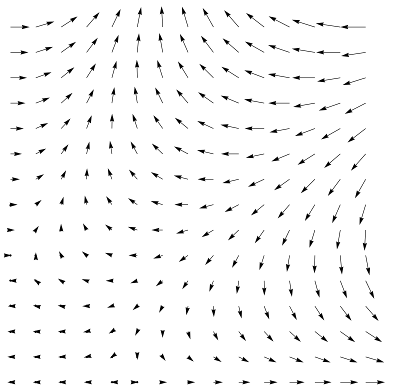

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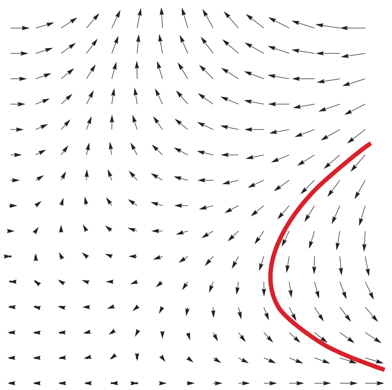
2. Start at initial state y_0 at initial time t_0
3. Follow the direction of the vector



Differential Equations as Models of Continuous Processes

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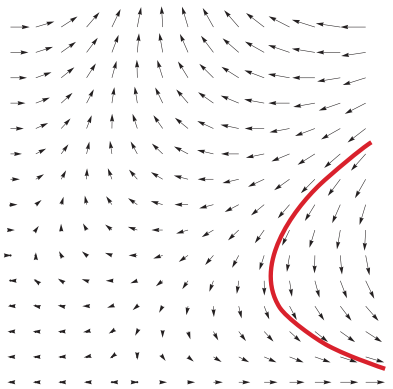
↯ The diagram should really show infinitely many vectors . . .



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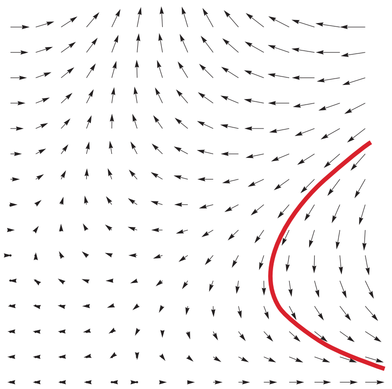

Point mass motion ODE: $x' = v, v' = a$

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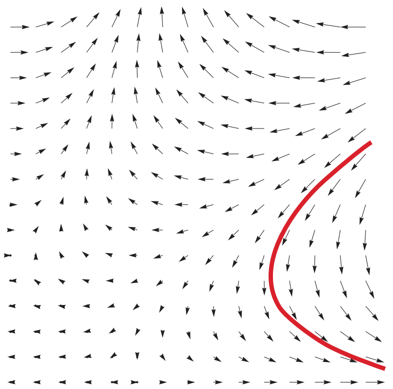
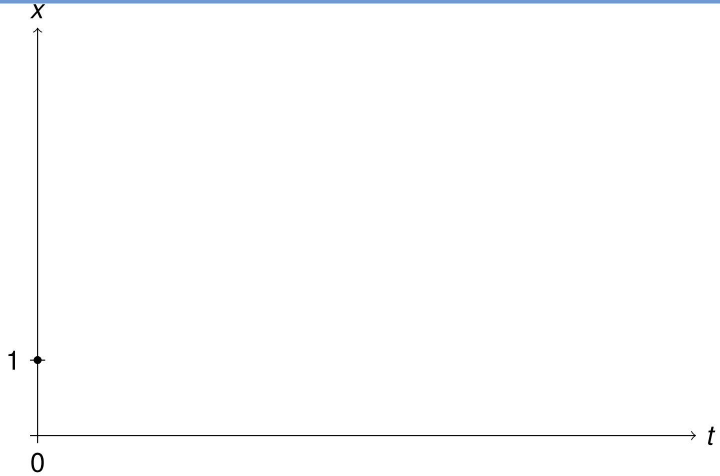


Point mass motion ODE: $x' = v, v' = a$

Newton's law of cooling ODE: $x' = k(T - x)$

Intuition for Differential Equations



$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix}$$

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$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x(t+\Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(0) := 1 \end{pmatrix}$$

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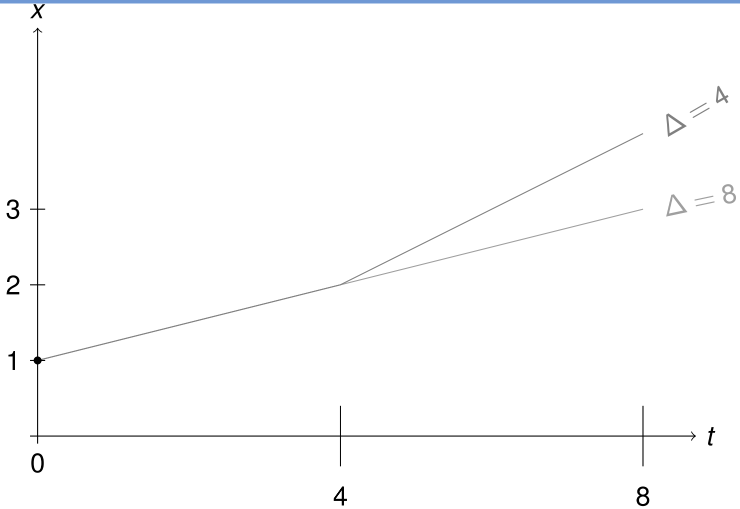

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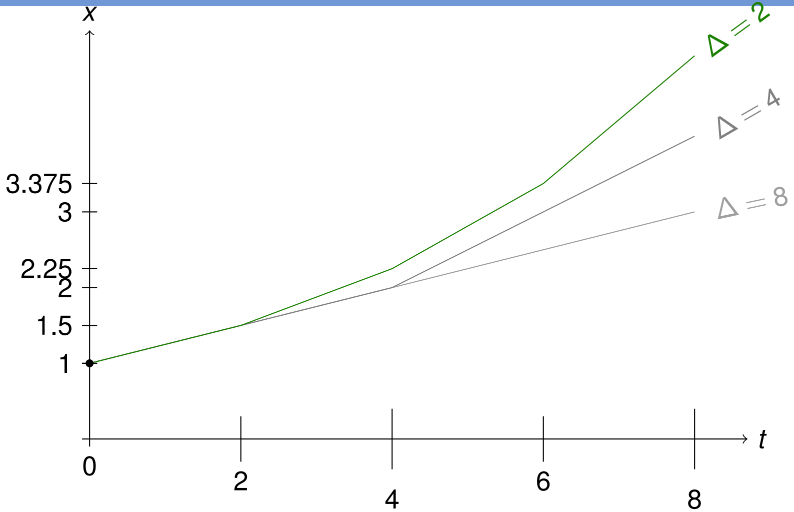

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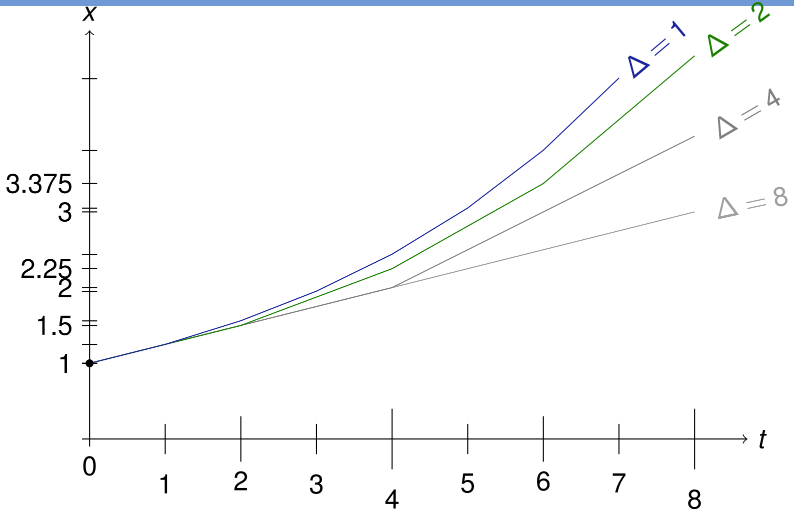

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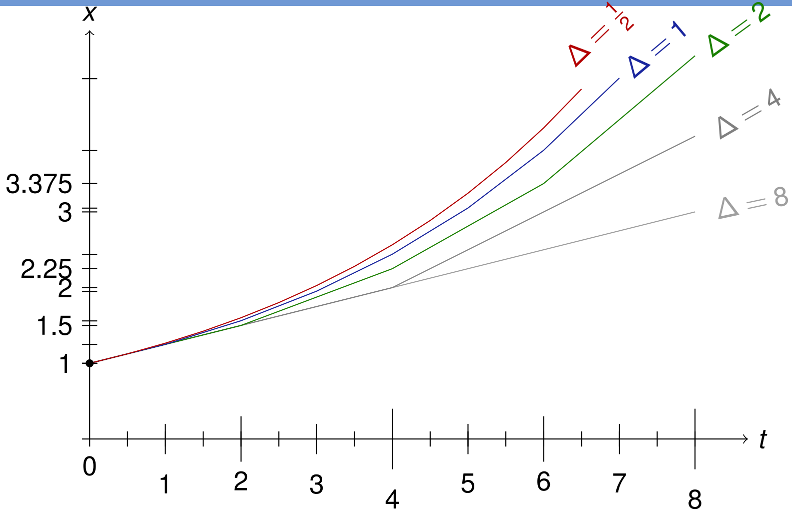

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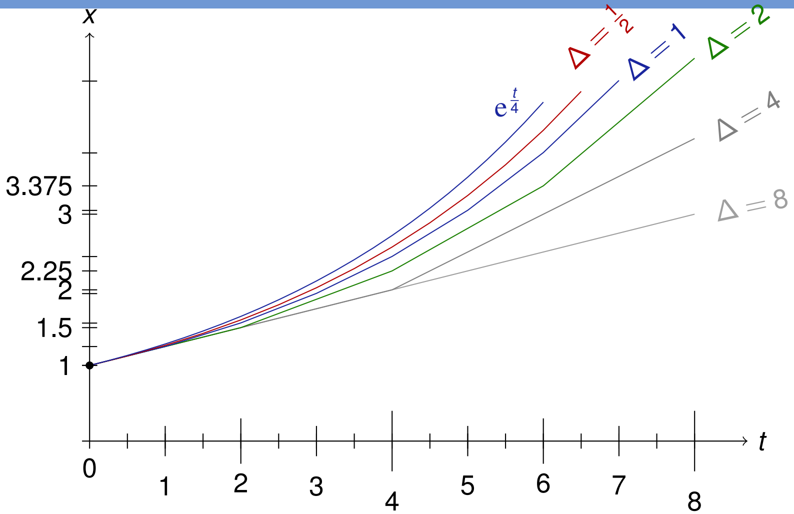

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Outline

The Meaning of Differential Equations

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at *every* point in space?
3. Could these directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure

$\rightsquigarrow$

We immediately want to get into the habit of
studying the behavior and exact meaning
of all relevant aspects of CPS.

Differential Equations & Initial Value Problems

Definition (Ordinary Differential Equation, ODE)

$f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected set). Then $Y : I \to \mathbb{R}^n$ is *solution* of initial value problem (IVP)

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.
If f continuous, then Y continuously differentiable.

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Example: A Constant Differential Equation

Example (Initial value problem)

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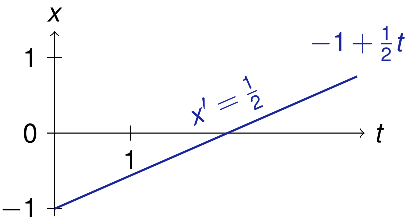

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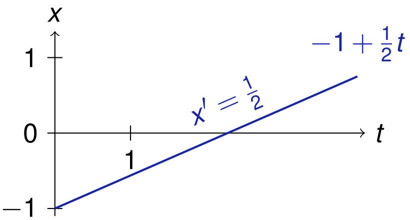
Check by inserting solution into ODE+IVP.

$$\begin{pmatrix} (x(t))' = (\frac{1}{2}t - 1)' = \frac{1}{2} \\ x(0) = \frac{1}{2} \cdot 0 - 1 = -1 \end{pmatrix}$$

□



Example: A Linear Differential Equation from before

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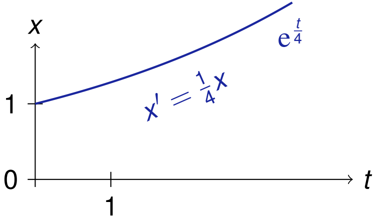

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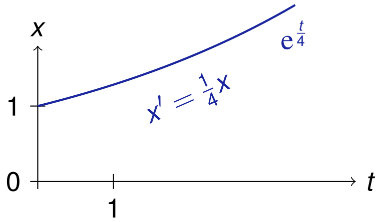

Example: Linear Dynamics

Example (Initial value problem)

$$\begin{pmatrix} v'(t) = w(t) \\ w'(t) = -v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix} \quad \text{has solution}$$

Example: Rotational Dynamics

Example (Initial value problem)

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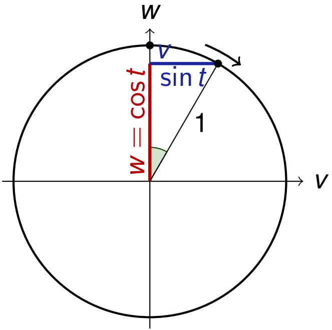
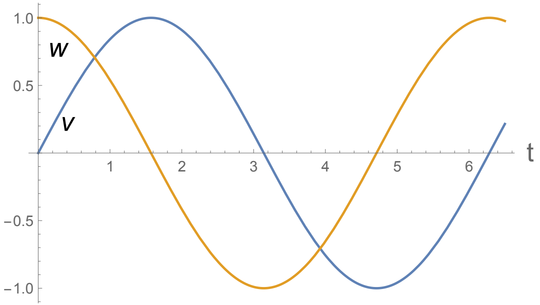
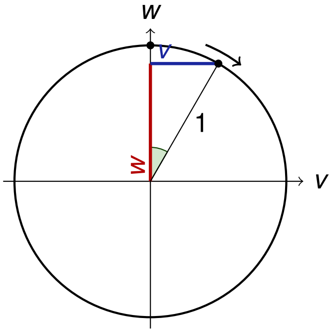


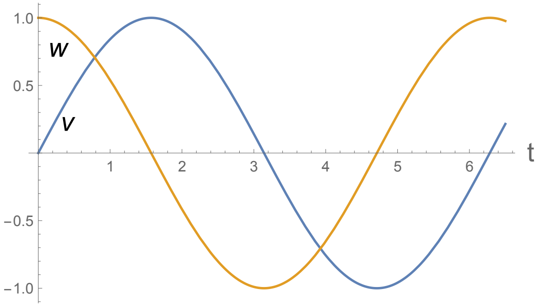


Example: Rotational Dynamics

Example (Initial value problem)

$$\begin{pmatrix} v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix}$$ has solution





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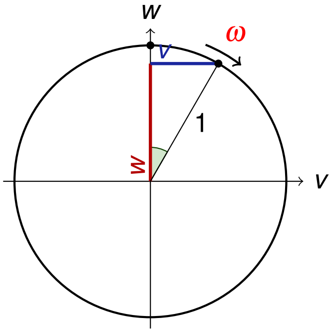

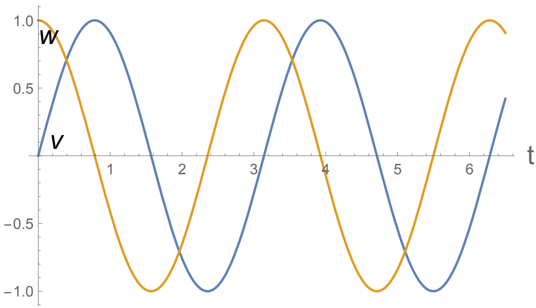

Example: More Dynamics

Example (Initial value problem)

$$\begin{pmatrix} x'(t) = v(t) \\ y'(t) = w(t) \\ v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ x(0) = x_0 \\ y(0) = y_0 \\ v(0) = v_0 \\ w(0) = w_0 \end{pmatrix}$$

Example: Planar Motion Dynamics

Example (Initial value problem)

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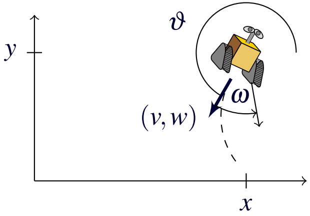

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

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Takeaway Message

Descriptive power of differential equations

1. Solutions of differential equations can be much more involved than the differential equations themselves.
2. Representational and descriptive power of differential equations!
3. Simple differential equations can describe quite complicated physical processes.
4. Local description as the direction into which the system evolves.

Outline

Evolution Domain Constraints

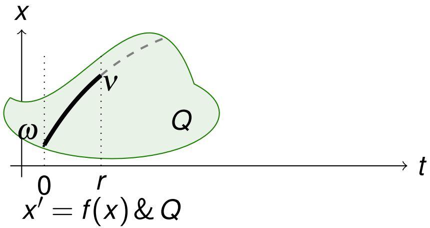
So far: follow vector field from initial values indefinitely
Now: enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A *differential equation $x' = f(x)$ with evolution domain Q* is denoted by

$$x' = f(x) \,\&\, Q$$

conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ **and** the evolution domain Q.



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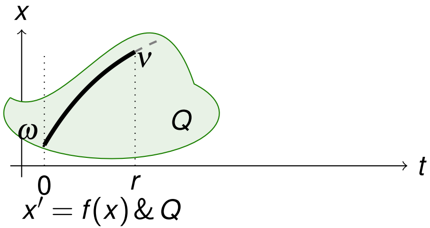

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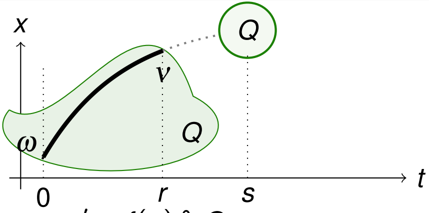

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$x' = v, v' = a, t' = 1 \,\&\, t \leq \varepsilon$	stops at clock ε at the latest
$x' = v, v' = a, t' = 1 \,\&\, v \geq 0$	stops before velocity negative
$x' = y, y' = x + y^2 \,\&\, true$	no constraint

Evolution Domain Constraints

So f… vector field from initial values ind…

…ble Cyber to interact with Physi…

Define: Terms

Define: Formulas

Definition (Evolution domain constraints)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

$$x' = f(x) \,\&\, Q$$

conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ **and** the evolution domain Q.

$x' = v, v' = a, t' = 1 \,\&\, t \le \varepsilon$	stops at clock ε at the latest
$x' = v, v' = a, t' = 1 \,\&\, v \ge 0$	stops before velocity negative
$x' = y, y' = x + y^2 \,\&\, true$	no constraint

Terms: Syntax

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

where $e, \tilde{e}$ are terms, $x \in \mathscr{V}$ is a variable, $c \in \mathbb{Q}$ a rational number constant

Terms: Syntax

Definition (Syntax of terms)

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Variable Constant Add Multiply

Terms: Syntax & Semantics

Definition (Syntax of terms)

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$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms) ($[\![\cdot]\!] : \mathrm{Trm} \to (\mathscr{S} \to \mathbb{R})$)

The *value of term* e in state $\omega : \mathscr{V} \to \mathbb{R}$ is a real number denoted $\omega[\![e]\!]$ and is defined by induction on the structure of e:

$\omega[\![x]\!] = \omega(x)$	if $x \in \mathscr{V}$ is a variable
$\omega[\![c]\!] = c$	if $c \in \mathbb{Q}$ is a rational constant
$\omega[\![e + \tilde{e}]\!] = \omega[\![e]\!] + \omega[\![\tilde{e}]\!]$	addition of reals
$\omega[\![e \cdot \tilde{e}]\!] = \omega[\![e]\!] \cdot \omega[\![\tilde{e}]\!]$	multiplication of reals

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

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$$\begin{aligned}
\omega[\![x]\!] &= \omega(x) && \text{if } x \in \mathscr{V} \text{ is a variable}\\
\omega[\![c]\!] &= c && \text{if } c \in \mathbb{Q} \text{ is a rational constant}\\
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\omega[\![e\cdot\tilde{e}]\!] &= \omega[\![e]\!] \cdot \omega[\![\tilde{e}]\!] && \text{multiplication of reals}
\end{aligned}$$

$\omega[\![4+x\cdot 2]\!] =$ if $\omega(x) = 5$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e,\tilde{e} ::= x \mid c \mid e+\tilde{e} \mid e\cdot\tilde{e}$$

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The *value of term* e in state $\omega : \mathscr{V} \to \mathbb{R}$ is a real number denoted $\omega\llbracket e\rrbracket$ and is defined by induction on the structure of e:

$$\omega\llbracket x\rrbracket = \omega(x) \quad \text{if } x \in \mathscr{V} \text{ is a variable}$$

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$$\omega\llbracket e+\tilde{e}\rrbracket = \omega\llbracket e\rrbracket + \omega\llbracket\tilde{e}\rrbracket \quad \text{addition of reals}$$

$$\omega\llbracket e\cdot\tilde{e}\rrbracket = \omega\llbracket e\rrbracket \cdot \omega\llbracket\tilde{e}\rrbracket \quad \text{multiplication of reals}$$

$$\omega\llbracket 4+x\cdot 2\rrbracket = \omega\llbracket 4\rrbracket + \omega\llbracket x\rrbracket \cdot \omega\llbracket 2\rrbracket = 4+\omega(x)\cdot 2 = 14 \quad \text{if } \omega(x)=5$$

Terms: Syntax & Semantics

Definition (Syntax of terms)

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\end{aligned}$$

What about $x - y$?

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

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\omega\llbracket e+\tilde{e}\rrbracket &= \omega\llbracket e\rrbracket + \omega\llbracket \tilde{e}\rrbracket && \text{addition of reals}\\
\omega\llbracket e\cdot\tilde{e}\rrbracket &= \omega\llbracket e\rrbracket \cdot \omega\llbracket \tilde{e}\rrbracket && \text{multiplication of reals}
\end{aligned}$$

What about $x - y$? Defined as $x + (-1) \cdot y$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

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What about x^4?

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term e* is a polynomial term defined by the grammar:

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Definition (Semantics of terms) $(\llbracket\cdot\rrbracket : \mathrm{Trm} \to (\mathscr{S} \to \mathbb{R}))$

The *value of term e* in state $\omega : \mathscr{V} \to \mathbb{R}$ is a real number denoted $\omega\llbracket e \rrbracket$ and is defined by induction on the structure of e:

$$\begin{aligned}
\omega\llbracket x \rrbracket &= \omega(x) && \text{if } x \in \mathscr{V} \text{ is a variable} \\
\omega\llbracket c \rrbracket &= c && \text{if } c \in \mathbb{Q} \text{ is a rational constant} \\
\omega\llbracket e + \tilde{e} \rrbracket &= \omega\llbracket e \rrbracket + \omega\llbracket \tilde{e} \rrbracket && \text{addition of reals} \\
\omega\llbracket e \cdot \tilde{e} \rrbracket &= \omega\llbracket e \rrbracket \cdot \omega\llbracket \tilde{e} \rrbracket && \text{multiplication of reals}
\end{aligned}$$

What about x^4? Defined as $x \cdot x \cdot x \cdot x$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms) $([\![\cdot]\!] : \mathrm{Trm} \to (\mathscr{S} \to \mathbb{R}))$

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What about x^n?

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms) $([\![\cdot]\!] : \mathrm{Trm} \to (\mathcal{S} \to \mathbb{R}))$

The *value of term* e in state $\omega : \mathcal{V} \to \mathbb{R}$ is a real number denoted $\omega[\![e]\!]$ and is defined by induction on the structure of e:

$$\omega[\![x]\!] = \omega(x) \qquad \text{if } x \in \mathcal{V} \text{ is a variable}$$

$$\omega[\![c]\!] = c \qquad \text{if } c \in \mathbb{Q} \text{ is a rational constant}$$

$$\omega[\![e + \tilde{e}]\!] = \omega[\![e]\!] + \omega[\![\tilde{e}]\!] \qquad \text{addition of reals}$$

$$\omega[\![e \cdot \tilde{e}]\!] = \omega[\![e]\!] \cdot \omega[\![\tilde{e}]\!] \qquad \text{multiplication of reals}$$

What about x^n? Defined as $x \cdot x \cdot x \cdot x \cdot x \cdot \ldots$, wait when do we stop???

First-Order Logic Formulas: Syntax

Definition (Syntax of first-order logic formulas)

The *formulas* of *FOL of real arithmetic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x\, P \mid \exists x\, P$$

First-Order Logic Formulas: Syntax

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The *formulas* of *FOL of real arithmetic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \to Q \mid P \leftrightarrow Q \mid \forall x\, P \mid \exists x\, P$$

Greater-or-equal | Not | And | Or | Imply | Equiv | All | Exists

First-Order Logic Formulas: Syntax & Semantics

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Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$	iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$
$\omega \models e \geq \tilde{e}$	iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$
$\omega \models \neg P$	iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$
$\omega \models P \land Q$	iff $\omega \models P$ and $\omega \models Q$
$\omega \models P \lor Q$	iff $\omega \models P$ or $\omega \models Q$
$\omega \models P \rightarrow Q$	iff $\omega \not\models P$ or $\omega \models Q$
$\omega \models \forall x\, P$	iff $\omega_x^d \models P$ for all $d \in \mathbb{R}$
$\omega \models \exists x\, P$	iff $\omega_x^d \models P$ for some $d \in \mathbb{R}$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

First-Order Logic Formulas: Syntax & Semantics

$\omega \models P$ formula P is true in state ω

$\vDash P$ formula P is *valid*, i.e., true in all states ω, i.e., $\omega \models P$ for all ω

$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$ iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$

$\omega \models e \geq \tilde{e}$ iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$

$\omega \models \neg P$ iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$

$\omega \models P \wedge Q$ iff $\omega \models P$ and $\omega \models Q$

$\omega \models P \vee Q$ iff $\omega \models P$ or $\omega \models Q$

$\omega \models P \rightarrow Q$ iff $\omega \not\models P$ or $\omega \models Q$

$\omega \models \forall x\, P$ iff $\omega_x^d \models P$ for all $d \in \mathbb{R}$

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First-Order Logic Formulas: Syntax & Semantics

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$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

$\exists y\,(y^2 \leq x)$ for $\omega(x) = 5$ and $\nu(x) = -5$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

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$\omega \models e \geq \tilde{e}$ iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$

$\omega \models \neg P$ iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$

$\omega \models P \wedge Q$ iff $\omega \models P$ and $\omega \models Q$

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$$\omega_x^d(y) = \begin{cases} d & \text{if } y{=}x \\ \omega(y) & \text{if } y{\neq}x \end{cases}$$

First-Order Logic Formulas: Syntax & Semantics

$\omega \models P$ formula P is true in state ω

$\vDash P$ formula P is *valid*, i.e., true in all states ω, i.e., $\omega \models P$ for all ω

$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

$\omega \models \exists y\, (y^2 \leq x)$ but $\nu \not\models \exists y\, (y^2 \leq x)$ for $\omega(x) = 5$ and $\nu(x) = -5$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$ iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$

$\omega \models e \geq \tilde{e}$ iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$

$\omega \models \neg P$ iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$

$\omega \models P \wedge Q$ iff $\omega \models P$ and $\omega \models Q$

$\omega \models P \vee Q$ iff $\omega \models P$ or $\omega \models Q$

$\omega \models P \rightarrow Q$ iff $\omega \not\models P$ or $\omega \models Q$

$\omega \models \forall x\, P$ iff $\omega_x^d \models P$ for all $d \in \mathbb{R}$

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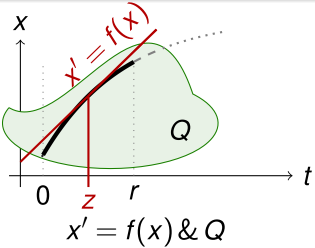
$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

Semantics of ODEs with Evolution Constraints

Definition (Semantics of differential equations)

A function $\varphi : [0,r] \to \mathscr{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \,\&\, Q$, written $\varphi \models x' = f(x) \land Q$, iff:

1. $\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x)$ and $\varphi(z) \models Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



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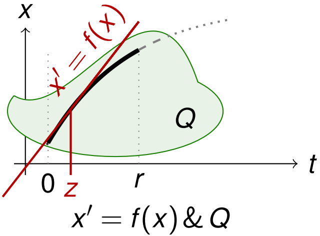

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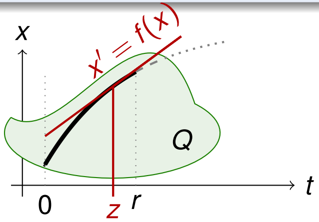


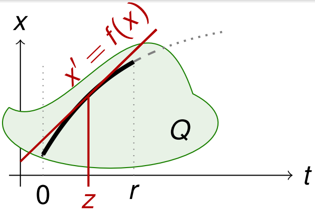
$x' = f(x) \,\&\, Q$

Semantics of ODEs with Evolution Constraints

Definition (Semantics of differential equations)

A function $\varphi : [0,r] \to \mathscr{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \,\&\, Q$, written $\varphi \models x' = f(x) \wedge Q$, iff:

1. $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x)$ and $\varphi(z) \models Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



$x' = f(x) \,\&\, Q$

Outline

Summary: Differential Equations & Domains

Definition (Syntax of terms)

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

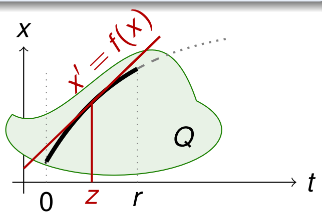
Definition (Syntax of first-order logic formulas)

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x\, P \mid \exists x\, P$$

Definition (Syntax of continuous programs)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

$$x' = f(x) \,\&\, Q$$



Further Reading I

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