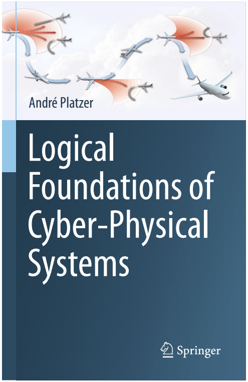
06: Truth & Proof

Logical Foundations of Cyber-Physical Systems



Stefan Mitsch

Carnegie Mellon University
Computer Science Department

Outline

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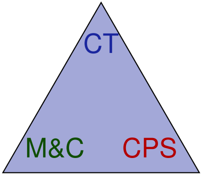
Learning Objectives
Truth & Proof

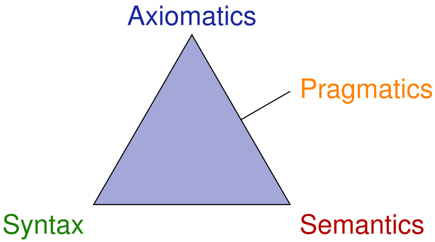
systematic reasoning for CPS
verifying CPS models at scale
pragmatics: how to use axiomatics to justify truth
structure of proofs and their arithmetic



discrete+continuous relation
with evolution domains

analytic skills for CPS

Logical Trinity with Extra Leg



- **Syntax** defines the notation
  What problems are we allowed to write down?
- **Semantics** what carries meaning.
  What real or mathematical objects does the syntax stand for?
- **Axiomatics** internalizes semantic relations into universal syntactic transformations.
- **Pragmatics** how to use axiomatics to justify syntactic rendition of semantical concepts. How to do a proof?

Outline

Sequent Calculus

Definition (Sequent)

$$\Gamma \vdash \Delta$$

has the same meaning as $\bigwedge_{P \in \Gamma} P \to \bigvee_{Q \in \Delta} Q$.
The *antecedent* Γ and *succedent* Δ are finite sets of dL formulas.

Definition (Soundness of sequent calculus proof rules)

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

is *sound* iff validity of all premises implies validity of conclusion:

$$\text{If } \vDash (\Gamma_1 \vdash \Delta_1) \text{ and } \dots \text{ and } \vDash (\Gamma_n \vdash \Delta_n) \text{ then } \vDash (\Gamma \vdash \Delta)$$

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construct proofs up $\uparrow$

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Definition (Soundness of sequent calculus proof rules)

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$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

$\downarrow$ validity transfers down

is *sound* iff validity of all premises implies validity of conclusion:

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Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{L}\ \frac{}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge L \quad \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

$\wedge$L: assume conjuncts separately
It successively handles all top-level $\wedge$ in assumptions but not nested in $A \vee (B \wedge C) \vdash C$ which needs rules for other propositional operators

Propositional Proof Rules of Sequent Calculus

$$\wedge R \ \frac{}{\Gamma \vdash P \wedge Q, \Delta}$$

$$\wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\text{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}$$

$$\wedge\text{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\text{R} \quad \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}$$

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$\wedge$R: prove conjuncts separately

Propositional Proof Rules of Sequent Calculus

$$\wedge R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \; \frac{}{\Gamma \vdash P \vee Q, \Delta}$$

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Propositional Proof Rules of Sequent Calculus

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Propositional Proof Rules of Sequent Calculus

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$$\wedge L \; \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

$\vee$R: split disjunctions in succedent where comma has a disjunctive meaning

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{}{\Gamma, P \vee Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

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$$\wedge\text{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\text{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

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$$\wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$\vee$L: handle disjunctive assumption by one proof for each assumed disjunct

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{}{\Gamma \vdash P \rightarrow Q, \Delta}$$

Propositional Proof Rules of Sequent Calculus

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$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

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Propositional Proof Rules of Sequent Calculus

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$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\to\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \to Q, \Delta}$$

$\to$R: prove implication by assuming LHS when proving RHS

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

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Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\to\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \to Q, \Delta}$$

$$\to\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \to Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$\rightarrow$L: assume RHS of an assumed implication after proving its LHS

Propositional Proof Rules of Sequent Calculus

$$\wedge R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \ \frac{}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R \ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow L \ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\mathrm{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

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$$\wedge R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \; \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \; \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \; \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \; \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R \; \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow L \; \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$\neg R$: prove $\neg P$ by proving contradiction (or Δ options) from assumption P

Propositional Proof Rules of Sequent Calculus

$$\wedge R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \; \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \; \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \; \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \; \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L \; \frac{}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \; \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow L \; \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\mathrm{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg\mathrm{L}\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

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$$\wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L \ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\rightarrow L \ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$\neg L$: assume $\neg P$ by proving its opposite P

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\mathrm{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg\mathrm{L}\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \mathrm{id}\ \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \; \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \; \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \; \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \; \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L \; \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \; \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \; \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow L \; \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

id: proof done (marked $*$) when succedent to prove is in antecedent

Propositional Proof Rules of Sequent Calculus

$$\wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id}\ \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

id: only way to finish a proof (in propositional logic!)

Propositional Proof Rules of Sequent Calculus

$$\land R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \land Q, \Delta} \qquad \lor R \ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \lor Q, \Delta} \qquad \neg R \ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\land L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \land Q \vdash \Delta} \qquad \lor L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \lor Q \vdash \Delta} \qquad \neg L \ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \ \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow L \ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut} \ \frac{}{\Gamma \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \; \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \; \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \; \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \; \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L \; \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \; \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \; \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow L \; \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut} \; \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\mathrm{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg\mathrm{L}\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \mathrm{id}\ \frac{}{\Gamma, P \vdash P, \Delta}$$

$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \mathrm{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

cut: Show lemma C and then assume lemma C

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\mathrm{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\mathrm{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\mathrm{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg\mathrm{L}\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \mathrm{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top\mathrm{R}\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \mathrm{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R \ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L \ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R \ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R \ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L \ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut} \ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

$\top$R: proof done (marked $*$) when proving trivial *true* (used rarely)

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$\top$R: what rule to use when *true* in antecedent?

Propositional Proof Rules of Sequent Calculus

$$\wedge\text{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\text{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg\text{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

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$$\rightarrow\text{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot\text{L}\ \frac{}{\Gamma, \mathit{false} \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

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$\bot$L: proof done (marked $*$) when assuming trivial *false* (used rarely)

Propositional Proof Rules of Sequent Calculus

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$\bot L$: what rule to use when *false* in succedent?

Sequent Proof Example (Simple)

$$\frac{}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}}{\vdash v^2\leq 10 \land b>0 \rightarrow b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\ {\rightarrow}\mathrm{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{}{v^2\le 10 \land b>0 \vdash b>0} \qquad \dfrac{}{v^2\le 10 \land b>0 \vdash \neg(v\ge 0) \lor v^2\le 10}}{\dfrac{v^2\le 10 \land b>0 \vdash b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}{\vdash v^2\le 10 \land b>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}\,{\to}\mathrm{R}}\,{\land}\mathrm{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{\dfrac{}{v^2 \le 10, b>0 \vdash b>0}}{v^2 \le 10 \land b>0 \vdash b>0}\,\land\mathrm{L} \qquad \dfrac{}{v^2 \le 10 \land b>0 \vdash \neg(v \ge 0) \lor v^2 \le 10}}{\dfrac{v^2 \le 10 \land b>0 \vdash b>0 \land (\neg(v \ge 0) \lor v^2 \le 10)}{\vdash v^2 \le 10 \land b>0 \to b>0 \land (\neg(v \ge 0) \lor v^2 \le 10)}\,\to\mathrm{R}}\,\land\mathrm{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{\dfrac{*}{v^2\le 10, {\color{red}b>0} \vdash {\color{red}b>0}}\,\text{id}}{v^2\le 10 \land b>0 \vdash b>0}\,\land\text{L} \qquad \dfrac{}{v^2\le 10 \land b>0 \vdash \neg(v\ge 0) \lor v^2\le 10}}{\dfrac{v^2\le 10 \land b>0 \vdash b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}{\vdash v^2\le 10 \land b>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}\,\to\text{R}}\,\land\text{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{\dfrac{*}{v^2\leq 10, b>0 \vdash b>0}\,\mathrm{id}}{v^2\leq 10 \land b>0 \vdash b>0}\,\land\mathrm{L} \qquad \dfrac{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0) \lor v^2\leq 10}\,\lor\mathrm{R}}{\dfrac{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}{\vdash v^2\leq 10 \land b>0 \rightarrow b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\rightarrow\mathrm{R}}\,\land\mathrm{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{
  \dfrac{
    \dfrac{*}{v^2\leq 10, b>0 \vdash b>0}\,\text{id}
  }{v^2\leq 10 \land b>0 \vdash b>0}\,\land\text{L}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{v^2\leq 10, b>0 \vdash \neg(v\geq 0), v^2\leq 10}
    }{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,\land\text{L}
  }{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0) \lor v^2\leq 10}\,\lor\text{R}
}{
  \dfrac{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\to\text{R}
}\,\land\text{R}
$$

Sequent Proof Example (Simple)

$$
\dfrac{\dfrac{\dfrac{*}{v^2\le 10, b>0 \vdash b>0}\,\text{id}}{v^2\le 10 \land b>0 \vdash b>0}\,\land\text{L} \qquad \dfrac{\dfrac{\dfrac{*}{\color{red}{v^2\le 10}, b>0 \vdash \neg(v\ge 0), \color{red}{v^2\le 10}}\,\text{id}}{v^2\le 10 \land b>0 \vdash \neg(v\ge 0), v^2\le 10}\,\land\text{L}}{v^2\le 10 \land b>0 \vdash \neg(v\ge 0) \lor v^2\le 10}\,\lor\text{R}}{\dfrac{v^2\le 10 \land b>0 \vdash b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}{\vdash v^2\le 10 \land b>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}\,\to\text{R}}\,\land\text{R}
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Soundness of Proof Rules

Lemma

$$\wedge R \quad \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \text{ is sound}$$

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Proof

using $[\![P \wedge Q]\!] = [\![P]\!] \cap [\![Q]\!]$.

WLOG: $\omega \in [\![G]\!]$ for all $G \in \Gamma$ and $\omega \notin [\![D]\!]$ for all $D \in \Delta$ (why?)
By premise: $\omega \in [\![\Gamma \vdash P, \Delta]\!]$ and $\omega \in [\![\Gamma \vdash Q, \Delta]\!]$
By WLOG: $\omega \in [\![P]\!]$ and $\omega \in [\![Q]\!]$ $\square$
By semantics: $\omega \in [\![P \wedge Q]\!]$
By definition: $\omega \in [\![\Gamma \vdash P \wedge Q, \Delta]\!]$

Soundness of dL

Theorem

dL *sequent calculus is sound: every* dL *formula with a proof is valid.*

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1. Sequent proof ends with some proof step:

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

The subproof of each premise $\Gamma_i \vdash \Delta_i$ is smaller, so $\vDash \Gamma_i \vdash \Delta_i$ by IH.
All dL proof rules are proved sound, also the one used above, i.e.:

$$\text{If } \vDash (\Gamma_1 \vdash \Delta_1) \text{ and } \dots \text{ and } \vDash (\Gamma_n \vdash \Delta_n) \text{ then } \vDash (\Gamma \vdash \Delta)$$

Thus, $\vDash (\Gamma \vdash \Delta)$. □

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Thus, $\vDash (\Gamma \vdash \Delta)$. □

Todo Always make sure *every* axiom and proof rule we adopt is sound!

Dynamics Proof Rules of Sequent Calculus

Have: Left and right proof rule for all propositional connectives

Need: Left and right proof rule for all top-level operators in all modalities?

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$$[\cup]R \ \frac{\Gamma \vdash [\alpha]P \land [\beta]P, \Delta}{\Gamma \vdash [\alpha \cup \beta]P, \Delta}$$

Boring! Already follow from the axiom

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

$$[\cup]L \ \frac{\Gamma, [\alpha]P \land [\beta]P \vdash \Delta}{\Gamma, [\alpha \cup \beta]P \vdash \Delta}$$

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Rules $[\cup]R$,$[\cup]L$ would only apply top-level, not in any other logical context such as

$$[x'' = -g]\_$$

$$[\cup] \ \frac{}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)}$$

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Contextual Equivalence: substituting equals for equals

$$\text{CER} \; \frac{\Gamma \vdash C(Q), \Delta \quad \vdash P \leftrightarrow Q}{\Gamma \vdash C(P), \Delta} \qquad \text{CEL} \; \frac{\Gamma, C(Q) \vdash \Delta \quad \vdash P \leftrightarrow Q}{\Gamma, C(P) \vdash \Delta}$$

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$$[?x{=}0; v := -cv \cup ?x{\geq}0]B(x,v) \leftrightarrow [?x{=}0; v := -cv]B(x,v) \land [?x{\geq}0]B(x,v)$$

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$$[\cup] \ \frac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

Simple Example Proof Dynamics in Sequent Calculus

$$\dfrac{}{\vdash [a := -b]\forall w\,\big(w^2 \le 10 \land -a > 0 \rightarrow b > 0 \land (\neg(w \ge 0) \lor w^2 \le 10)\big)}\,[:=]$$

Simple Example Proof Dynamics in Sequent Calculus

$$[a := -b]\forall w\big(w^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(w \geq 0) \lor v^2 \leq 10)\big) \leftrightarrow$$
$$\forall w\big(w^2 \leq 10 \land -(-b) > 0 \rightarrow b > 0 \land (\neg(w \geq 0) \lor w^2 \leq 10)\big) \text{ by } [:=]$$

$$\cfrac{\overline{\vdash \forall w\big(w^2 \leq 10 \land -(-b) > 0 \rightarrow b > 0 \land (\neg(w \geq 0) \lor w^2 \leq 10)\big)}}{\vdash [a := -b]\forall w\big(w^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(w \geq 0) \lor w^2 \leq 10)\big)}\,[:=]$$

Our earlier propositional proof:

$$
\begin{prooftree}
\AxiomC{$*$}
\LeftLabel{id}
\UnaryInfC{$v^2\le10, b>0 \vdash b>0$}
\LeftLabel{$\wedge$L}
\UnaryInfC{$v^2\le10 \wedge b>0 \vdash b>0$}
\AxiomC{$*$}
\LeftLabel{id}
\UnaryInfC{$v^2\le10 \vdash , b>0 \neg(v\ge0), v^2\le10$}
\LeftLabel{$\wedge$L}
\UnaryInfC{$v^2\le10 \wedge b>0 \vdash \neg(v\ge0), v^2\le10$}
\LeftLabel{$\vee$R}
\UnaryInfC{$v^2\le10 \wedge b>0 \vdash \neg(v\ge0) \vee v^2\le10$}
\LeftLabel{$\wedge$R}
\BinaryInfC{$v^2\le10 \wedge b>0 \vdash b>0 \wedge (\neg(v\ge0) \vee v^2\le10)$}
\LeftLabel{$\rightarrow$R}
\UnaryInfC{$\vdash v^2\le10 \wedge b>0 \rightarrow b>0 \wedge (\neg(v\ge0) \vee v^2\le10)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash \forall w \big(w^2\le10 \wedge -(-b)>0 \rightarrow b>0 \wedge (\neg(w\ge0) \vee w^2\le10)\big)$}
\LeftLabel{$[:=]$}
\UnaryInfC{$\vdash [a:=-b]\forall w \big(w^2\le10 \wedge -a>0 \rightarrow b>0 \wedge (\neg(w\ge0) \vee w^2\le10)\big)$}
\end{prooftree}
$$

Our earlier propositional proof:

$$
\mathrm{id}\ \dfrac{*}{v^2\leq 10 \vdash , b>0 \neg(v\geq 0), v^2\leq 10}
$$

$$
\land\mathrm{R}\ \dfrac{
\land\mathrm{L}\ \dfrac{\mathrm{id}\ \dfrac{*}{v^2\leq 10, b>0 \vdash b>0}}{v^2\leq 10 \land b>0 \vdash b>0}
\qquad
\lor\mathrm{R}\ \dfrac{\land\mathrm{L}\ \dfrac{v^2\leq 10 \vdash , b>0 \neg(v\geq 0), v^2\leq 10}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0) \lor v^2\leq 10}
}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}
$$

$$
\to\mathrm{R}\ \dfrac{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}
$$

$$
[:=]\ \dfrac{\dfrac{}{\vdash \forall w\big(w^2\leq 10 \land -(-b)>0 \to b>0 \land (\neg(w\geq 0) \lor w^2\leq 10)\big)}}{\vdash [a:=-b]\forall w\big(w^2\leq 10 \land -a>0 \to b>0 \land (\neg(w\geq 0) \lor w^2\leq 10)\big)}
$$

Need to reason about real arithmetic

Here: to glue previous propositional proof with this dynamic proof

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{}{\Gamma \vdash \forall x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall R \;\; \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\text{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$\forall$R: show for fresh variable y about which we can't know anything

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\exists\mathrm{R}\ \frac{}{\Gamma \vdash \exists x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\exists\mathrm{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\text{R} \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\exists\text{R} \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$\exists$R: enough to show for any witness term e

Quantifier Proof Rules

$$\forall R \; \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \; \frac{}{\Gamma, \forall x\, p(x) \vdash \Delta}$$

$$\exists R \; \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

Quantifier Proof Rules

$$\forall R \; \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \; \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}$$

$$\exists R \; \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

Quantifier Proof Rules

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad \text{(arbitrary term } e\text{)}$$

$$\exists R \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

$\forall$L: even holds for arbitrary term e

Quantifier Proof Rules

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad \text{(arbitrary term } e\text{)}$$

$$\exists R \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

$$\exists L \ \frac{}{\Gamma, \exists x\, p(x) \vdash \Delta}$$

Quantifier Proof Rules

$$\forall R \quad \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \not\in \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \quad \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad \text{(arbitrary term } e\text{)}$$

$$\exists R \quad \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

$$\exists L \quad \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}$$

Quantifier Proof Rules

$$\forall R \quad \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\forall L \quad \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists R \quad \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists L \quad \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} \quad (y \notin \Gamma, \Delta, \exists x\, p(x))$$

$\exists$L: assume for fresh variable y about which we can't know anything

Quantifier Proof Rules

$$\forall R \quad \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L \quad \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists R \quad \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists L \quad \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} \quad (y \notin \Gamma,\Delta,\exists x\, p(x))$$

Important: soundness means that conclusion valid if all premises valid.

$$\frac{}{\vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\ {\to}\mathrm{R}$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$
\dfrac{[;]\ \dfrac{}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}}{\vdash A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\ \rightarrow\mathrm{R}
$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$A \vdash \forall t{\geq}0\, \big((H-\tfrac{g}{2}t^2{=}0 \to B(H-\tfrac{g}{2}t^2, -c(-gt))) \land (H-\tfrac{g}{2}t^2{\geq}0 \to B(H-\tfrac{g}{2}t^2, -g$$

$$[:=]\; A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2]\big((x{=}0 \to B(x, -c(-gt))) \land (x{\geq}0 \to B(x, -gt))\big)$$

$$[:=]\; A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big)$$

$$[;]\; A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big)$$

$$['] \; A \vdash [x'' = -g]\big((x = 0 \to B(x, -cv)) \land (x \geq 0 \to B(x, v))\big)$$

$$[:=]\; A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x, v)) \land (x \geq 0 \to B(x, v))\big)$$

$$[?]\; A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)$$

$$[;]\; A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)$$

$$[\cup]\; A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)$$

$$[;]\; A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)$$

$$\to\!R\; \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)$$

$$A \stackrel{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x, v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$[x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v) \leftrightarrow$
$[x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)$ by [;]

$$\dfrac{\dfrac{\dfrac{}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}[\cup]}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}[;]}{\vdash A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}{\rightarrow}\mathrm{R}$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[?x=0;v:=-cv\cup ?x\geq 0]B(x,v)\leftrightarrow$$
$$\big([?x=0;v:=-cv]B(x,v)\wedge[?x\geq 0]B(x,v)\big)\text{ by }[\cup]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash[x''=-g]\big([?x=0;v:=-cv]B(x,v)\wedge[?x\geq 0]B(x,v)\big)}}{A\vdash[x''=-g][?x=0;v:=-cv\cup ?x\geq 0]B(x,v)}[;]}{A\vdash[x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v)}[\cup]}{\vdash A\rightarrow[x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v)}[;]\ \rightarrow\mathrm{R}$$

$$A\stackrel{\text{def}}{\equiv}0\leq x\wedge x=H\wedge v=0\wedge g>0\wedge 1\geq c\geq 0$$

$$B(x,v)\stackrel{\text{def}}{\equiv}0\leq x\wedge x\leq H$$

$$\{x''=-g\}\stackrel{\text{def}}{\equiv}\{x'=v,v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$[?x = 0; v := -cv]B(x,v) \leftrightarrow$
$[?x = 0][v := -cv]B(x,v)$ by [;]

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)}[?]}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}[;]}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}[\cup]}{\vdash A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}[;]
$$

$\rightarrow$R

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[?x=0][v:=-cv]B(x,v) \leftrightarrow$$
$$x=0 \rightarrow [v:=-cv]B(x,v) \text{ by } [?]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash [x''=-g]\big((x=0 \rightarrow [v:=-cv]B(x,v)) \wedge (x \geq 0 \rightarrow B(x,v))\big)}[:=]}{A \vdash [x''=-g]\big([?x=0][v:=-cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}[?]}{A \vdash [x''=-g]\big([?x=0;v:=-cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}[;]}{A \vdash [x''=-g][?x=0;v:=-cv \cup ?x \geq 0]B(x,v)}[\cup]}{\dfrac{A \vdash [x''=-g;(?x=0;v:=-cv \cup ?x \geq 0)]B(x,v)}{\vdash A \rightarrow [x''=-g;(?x=0;v:=-cv \cup ?x \geq 0)]B(x,v)}\rightarrow R}[;]$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x=H \wedge v=0 \wedge g>0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$
$$\{x''=-g\} \stackrel{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$[v := -cv]B(x,v) \leftrightarrow$
$B(x,-cv)$ by $[:=]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash [x''=-g]\big((x=0 \to B(x,-cv)) \land (x\geq 0 \to B(x,v))\big)}[']}{A \vdash [x''=-g]\big((x=0 \to [v:=-cv]B(x,v)) \land (x\geq 0 \to B(x,v))\big)}[:=]}{A \vdash [x''=-g]\big([?x=0][v:=-cv]B(x,v) \land [?x\geq 0]B(x,v)\big)}[?]}{A \vdash [x''=-g]\big([?x=0;v:=-cv]B(x,v) \land [?x\geq 0]B(x,v)\big)}[;]}{A \vdash [x''=-g][?x=0;v:=-cv \cup ?x\geq 0]B(x,v)}[\cup]}{A \vdash [x''=-g;(?x=0;v:=-cv \cup ?x\geq 0)]B(x,v)}[;]}{\vdash A \to [x''=-g;(?x=0;v:=-cv \cup ?x\geq 0)]B(x,v)}{\to}R
$$

$$A \stackrel{\text{def}}{\equiv} 0\leq x \land x=H \land v=0 \land g>0 \land 1\geq c\geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x''=-g\} \stackrel{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$['] \quad [x' = f(x)]p(x) \leftrightarrow \forall t{\ge}0\, [x := y(t)]p(x)$$

$$
\begin{array}{ll}
{[;]} & \overline{A \vdash \forall t{\ge}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\ge}0 \to B(x,v))\big)} \\
{[']} & \overline{A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \ge 0 \to B(x,v))\big)} \\
{[:=]} & \overline{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \ge 0 \to B(x,v))\big)} \\
{[?]} & \overline{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \ge 0]B(x,v)\big)} \\
{[;]} & \overline{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \ge 0]B(x,v)\big)} \\
{[\cup]} & \overline{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \ge 0]B(x,v)} \\
{[;]} & \overline{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \ge 0)]B(x,v)} \\
{\to}\mathrm{R} & \overline{\vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \ge 0)]B(x,v)}
\end{array}
$$

$$A \stackrel{\text{def}}{\equiv} 0{\le}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\ge}c{\ge}0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \le x \land x \le H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$\begin{array}{ll}
\text{[:=]} & A \vdash \forall t{\geq}0\,[x := H-\frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \hline
\text{[;]} & A \vdash \forall t{\geq}0\,[x := H-\frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \hline
\text{[']} & A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
\text{[:=]} & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
\text{[?]} & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
\text{[;]} & A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
\text{[}\cup\text{]} & A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\ \hline
\text{[;]} & A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v) \\ \hline
\to\text{R} & \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \stackrel{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash \forall t{\geq}0\,[x := H-\frac{g}{2}t^2]\big((x{=}0\to B(x,-c(-gt)))\land(x{\geq}0\to B(x,-gt))\big)}[{:=}]}{A \vdash \forall t{\geq}0\,[x := H-\frac{g}{2}t^2][{\color{red}v := -gt}]\big((x{=}0\to B(x,-c{\color{red}v}))\land(x{\geq}0\to B(x,v))\big)}[{:=}]}{A \vdash \forall t{\geq}0\,[x := H-\frac{g}{2}t^2; v := -gt]\big((x{=}0\to B(x,-cv))\land(x{\geq}0\to B(x,v))\big)}[;]}{A \vdash [x''=-g]\big((x=0\to B(x,-cv))\land(x\geq 0\to B(x,v))\big)}[']}{A \vdash [x''=-g]\big((x=0\to [v:=-cv]B(x,v))\land(x\geq 0\to B(x,v))\big)}[{:=}]}{A \vdash [x''=-g]\big([?x=0][v:=-cv]B(x,v)\land[?x\geq 0]B(x,v)\big)}[?]}{A \vdash [x''=-g]\big([?x=0;v:=-cv]B(x,v)\land[?x\geq 0]B(x,v)\big)}[;]}{A \vdash [x''=-g][?x=0;v:=-cv\cup ?x\geq 0]B(x,v)}[\cup]}{A \vdash [x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v)}[;]}{\vdash A \to [x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v)}\to\!R
$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x''=-g\} \overset{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$
\begin{array}{rl}
& A \vdash \forall t{\geq}0\, \big((H{-}\frac{g}{2}t^2{=}0 \to B(H{-}\frac{g}{2}t^2, -c(-gt))) \land (H{-}\frac{g}{2}t^2{\geq}0 \to B(H{-}\frac{g}{2}t^2, -g \\
\hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2]\big((x{=}0 \to B(x, -c(-gt))) \land (x{\geq}0 \to B(x, -gt))\big) \\
\hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big) \\
\hline
[;] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big) \\
\hline
['] & A \vdash [x'' = -g]\big((x = 0 \to B(x, -cv)) \land (x \geq 0 \to B(x, v))\big) \\
\hline
[:=] & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x, v)) \land (x \geq 0 \to B(x, v))\big) \\
\hline
[?] & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big) \\
\hline
[;] & A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big) \\
\hline
[\cup] & A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v) \\
\hline
[;] & A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v) \\
\hline
\to\mathrm{R} & \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)
\end{array}
$$

$$A \stackrel{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x, v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$A \vdash \forall t{\geq}0\, \big((H{-}\tfrac{g}{2}t^2{=}0 \to B(H{-}\tfrac{g}{2}t^2, -c(-gt))) \land (H{-}\tfrac{g}{2}t^2{\geq}0 \to B(H{-}\tfrac{g}{2}t^2, -g$$

[:=]
$$A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2]\big((x{=}0 \to B(x, -c(-gt))) \land (x{\geq}0 \to B(x, -gt))\big)$$

[:=]
$$A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big)$$

[;]
$$A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big)$$

[′]
$$A \vdash [x'' = -g]\big((x = 0 \to B(x, -cv)) \land (x \geq 0 \to B(x, v))\big)$$

[:=]
$$A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x, v)) \land (x \geq 0 \to B(x, v))\big)$$

[?]
$$A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)$$

[;]
$$A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)$$

[∪]
$$A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)$$

[;]
$$A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)$$

→R
$$\vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)$$

$$A \stackrel{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x, v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

Outline

Real Arithmetic

Lemma ($\mathbb{R}$ real arithmetic)

$\mathrm{FOL}_{\mathbb{R}}$ *decidable, so side condition implementable:*

$$\mathbb{R}\ \frac{}{\Gamma \vdash \Delta} \qquad \left(\text{if } \bigwedge_{P\in\Gamma} P \to \bigvee_{Q\in\Delta} Q \text{ is valid in } \mathrm{FOL}_{\mathbb{R}}\right)$$

$$\mathbb{R}\frac{}{a > 0, b > 0 \vdash y \geq 0 \to ax^2 + by \geq 0} \qquad \mathbb{R}\frac{}{x^2 > 0 \vdash x > 0}$$

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Theorem (Tarski's quantifier elimination)

FOL$_{\mathbb{R}}$ *admits quantifier elimination: there is an algorithm that computes a quantifier-free formula* $\mathrm{QE}(P)$, *for each first-order real arithmetic formula* P, *that is equivalent, i.e.,* $P \leftrightarrow \mathrm{QE}(P)$ *is valid.*

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What if there are no quantifiers?

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What if there are no quantifiers? Universal closure with i$\forall$ $\dfrac{\Gamma \vdash \forall x\, P, \Delta}{\Gamma \vdash P, \Delta}$

Quantifier Elimination After Universal Closure

$$\forall R \frac{}{\vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)}$$

Quantifier Elimination After Universal Closure

$$\forall R \; \frac{}{\vdash \forall d \, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d] \, x \geq 0\big)}$$

Not a $FOL_{\mathbb{R}}$ formula so Tarski's quantifier elimination not applicable.

Quantifier Elimination After Universal Closure

$$\dfrac{\dfrac{}{\vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0}\,[\cup]}{\vdash \forall \textcolor{red}{d}\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)}\,\forall\mathrm{R}$$

Quantifier Elimination After Universal Closure

$$\dfrac{\dfrac{\dfrac{}{\vdash d \geq -x \to [x:=0]\,x \geq 0 \land [x:=x+d]\,x \geq 0}\,[:=]}{\vdash d \geq -x \to [x:=0 \cup x:=x+d]\,x \geq 0}\,[\cup]}{\vdash \forall d\,\big(d \geq -x \to [x:=0 \cup x:=x+d]\,x \geq 0\big)}\,\forall R$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x+d]\, x \geq 0}\,[:=]}{\vdash d \geq -x \rightarrow [x := 0]\, x \geq 0 \land [x := x+d]\, x \geq 0}\,[:=]}{\vdash d \geq -x \rightarrow [x := 0 \cup x := x+d]\, x \geq 0}\,[\cup]}{\vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x+d]\, x \geq 0\big)}\,\forall\mathrm{R}
$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0}\text{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x := 0]\,x \geq 0 \land [x := x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x := 0 \cup x := x+d]\,x \geq 0}[\cup]}{\vdash \forall d\,\big(d \geq -x \rightarrow [x := 0 \cup x := x+d]\,x \geq 0\big)}\forall\text{R}
$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \forall d\,\big(d \geq -x \to 0 \geq 0 \land x+d \geq 0\big)}\,\mathrm{i}\forall}{\vdash d \geq -x \to 0 \geq 0 \land x+d \geq 0}\,\mathrm{i}\forall}{\vdash d \geq -x \to 0 \geq 0 \land [x := x+d]\,x \geq 0}\,[:=]}{\vdash d \geq -x \to [x := 0]\,x \geq 0 \land [x := x+d]\,x \geq 0}\,[:=]}{\vdash d \geq -x \to [x := 0 \cup x := x+d]\,x \geq 0}\,[\cup]}{\vdash \forall d\,\big(d \geq -x \to [x := 0 \cup x := x+d]\,x \geq 0\big)}\,\forall R
$$

Quantifier Elimination After Universal Closure

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{R}\;\; \vdash \forall x \forall d \left(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\right)}{\vdash \forall d \left(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\right)}\text{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0}\text{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x + d]\, x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x := 0]\, x \geq 0 \land [x := x + d]\, x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0}[\cup]}{\vdash \forall d \left(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\right)}\forall\text{R}
$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash \forall x \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big)}\mathbb{R}}{\vdash \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big)}\mathrm{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0}\mathrm{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x:=x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x:=0]\,x \geq 0 \land [x:=x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x:=0 \cup x:=x+d]\,x \geq 0}[\cup]}{\vdash \forall d\, \big(d \geq -x \rightarrow [x:=0 \cup x:=x+d]\,x \geq 0\big)}\forall\mathrm{R}
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$$

We could also leave $\forall d$ alone and use axioms in the middle of the formula.

Quantifier Elimination After Universal Closure

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash \forall x \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big)}\mathbb{R}}{\vdash \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big)}\text{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0}\text{i}\forall}{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x:=x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x:=0]\,x \geq 0 \land [x:=x+d]\,x \geq 0}[:=]}{\vdash d \geq -x \rightarrow [x:=0 \cup x:=x+d]\,x \geq 0}[\cup]}{\vdash \forall d\, \big(d \geq -x \rightarrow [x:=0 \cup x:=x+d]\,x \geq 0\big)}\forall\text{R}
$$

Already use rule $\mathbb{R}$ for valid $\text{FOL}_{\mathbb{R}}$ formulas with free variables before i$\forall$

Instantiating Real-Arithmetic Quantifiers

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\dots) \quad \exists R \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\dots)$$

$$\forall L \ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\dots) \quad \exists L \ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\dots)$$

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$$['] \frac{\dfrac{}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(y(s))) \rightarrow [x := y(t)]P\big)}}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]P}$$

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$$

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$$
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\dfrac{*}{t\geq 0 \vdash 0\leq t\leq t, [x := y(t)]P}\ \mathbb{R}
\qquad
\dfrac{\dots}{\Gamma, t\geq 0, q(y(t)) \vdash [x := y(t)]P}
}{\Gamma, t\geq 0, 0\leq t\leq t \to q(y(t)) \vdash [x := y(t)]P}\ \to L
}{\Gamma, t\geq 0, \forall 0\leq s\leq t\, q(y(s)) \vdash [x := y(t)]P}\ \forall L
}{\Gamma, t\geq 0 \vdash (\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P}\ \to R
}{\Gamma \vdash t\geq 0 \to \big((\forall 0\leq s\leq t\, Q(y(s))) \to [x := y(t)]P\big)}\ \to R
}{\Gamma \vdash \forall t\geq 0\, \big((\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P\big)}\ \forall R
}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]P}\ [']
$$

Instantiating Real-Arithmetic Quantifiers

Derived Rule

$$\dfrac{\Gamma, t\geq 0, q(y(t)) \vdash [x:=y(t)]P}{\Gamma \vdash [x'=f(x)\,\&\,q(x)]P} \qquad (y'(t)=f(y))$$

$$
{\scriptstyle [']}\dfrac{{\scriptstyle \forall R}\dfrac{{\scriptstyle \to R}\dfrac{{\scriptstyle \to R}\dfrac{{\scriptstyle \forall L}\dfrac{{\scriptstyle \to L}\dfrac{{\scriptstyle \mathbb{R}}\dfrac{*}{t\geq0 \vdash 0\leq t\leq t, [x:=y(t)]P} \qquad \dfrac{\dots}{\Gamma, t\geq0, q(y(t)) \vdash [x:=y(t)]P}}{\Gamma, t\geq0, 0\leq {\color{red}t}\leq t\to q(y({\color{red}t})) \vdash [x:=y(t)]P}}{\Gamma, t\geq0, \forall 0\leq {\color{red}s}\leq t\, q(y({\color{red}s})) \vdash [x:=y(t)]P}}{\Gamma, t\geq0 \vdash (\forall 0\leq s\leq t\, q(y(s)))\to[x:=y(t)]P}}{\Gamma \vdash t\geq0\to\big((\forall 0\leq s\leq t\, Q(y(s)))\to[x:=y(t)]P\big)}}{\Gamma \vdash \forall t\geq0\,\big((\forall 0\leq s\leq t\, q(y(s)))\to[x:=y(t)]P\big)}}{\Gamma \vdash [x'=f(x)\,\&\,q(x)]P}
$$

Derived rule: rule that can be proved using other proof rules.

Weakening by Removing Assumptions

$$\text{WR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta}$$

$$\text{WL}\ \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}$$

$$\text{WL}\ \frac{r \geq 0 \vdash 0 \leq r \leq r}{A, r \geq 0 \vdash 0 \leq r \leq r}$$

Throw big arithmetic distraction A away by weakening since the proof is independent of formula A.

Occam's assumption razor

Think how hard it would be to prove a theorem with all the facts in all books of mathematics as assumptions.
Compared to a proof from just the two facts that matter.

Abbreviating Terms to Reduce Complexity

$$a \geq 0, t \geq 0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d \leq 8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

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Proof rules introducing such new variables will be studied in Chapter 12

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Proof rules introducing such new variables will be studied in Chapter 12
Inverse of a derived rule that turns assignments into equations:

$$[:=]_= \quad \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$$

Abbreviating Terms to Reduce Complexity

$$a\geq 0, t\geq 0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d\leq 8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

Proof rules introducing such new variables will be studied in Chapter 12
Inverse of a derived rule that turns assignments into equations:

$$[:=]_= \quad \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta} \quad (y \text{ new})$$

Creatively Cutting to Transform Questions

$$=\!\mathrm{R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!\mathrm{L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{}{(x-y)^2 \le 0, p(y) \vdash p(x)}\text{cut}}{(x-y)^2 \le 0 \land p(y) \vdash p(x)}\land\mathrm{L}}{\vdash (x-y)^2 \le 0 \land p(y) \to p(x)}\to\mathrm{R}$$

Creatively Cutting to Transform Questions

$$=\!R\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!L\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{\text{WL}\,\overline{(x-y)^2\leq 0, p(y) \vdash x = y, p(x)} \qquad \text{WL}\,\overline{(x-y)^2\leq 0, p(y), x = y \vdash p(x)}}{(x-y)^2\leq 0, p(y) \vdash p(x)}\ \text{cut}}{(x-y)^2\leq 0 \land p(y) \vdash p(x)}\ \land\text{L}}{\vdash (x-y)^2\leq 0 \land p(y) \to p(x)}\ \to\text{R}$$

Creatively Cutting to Transform Questions

$$=\!\mathrm{R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!\mathrm{L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\rightarrow\!\mathrm{R}\ \dfrac{\wedge\mathrm{L}\ \dfrac{\mathrm{cut}\ \dfrac{\mathrm{WL}\ \dfrac{\mathrm{WR}\ \dfrac{\mathbb{R}\ \dfrac{*}{(x-y)^2 \le 0 \vdash x = y}}{(x-y)^2 \le 0 \vdash x = y, p(x)}}{(x-y)^2 \le 0, p(y) \vdash x = y, p(x)} \qquad \mathrm{WL}\ \dfrac{}{(x-y)^2 \le 0, p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y) \vdash p(x)}}{(x-y)^2 \le 0 \wedge p(y) \vdash p(x)}}{\vdash (x-y)^2 \le 0 \wedge p(y) \rightarrow p(x)}$$

Creatively Cutting to Transform Questions

$$
{=}\mathrm{R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}
\qquad
{=}\mathrm{L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}
$$

$$
\rightarrow\mathrm{R}\ \cfrac{\wedge\mathrm{L}\ \cfrac{\mathrm{cut}\ \cfrac{\mathrm{WL}\ \cfrac{\mathrm{WR}\ \cfrac{\mathbb{R}\ \cfrac{*}{(x-y)^2 \le 0 \vdash x = y}}{(x-y)^2 \le 0 \vdash x = y, p(x)}}{(x-y)^2 \le 0, p(y) \vdash x = y, p(x)} \qquad \mathrm{WL}\ \cfrac{{=}\mathrm{R}\ \cfrac{}{p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y) \vdash p(x)}}{(x-y)^2 \le 0 \wedge p(y) \vdash p(x)}}{\vdash (x-y)^2 \le 0 \wedge p(y) \rightarrow p(x)}
$$

Creatively Cutting to Transform Questions

$$\text{=R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$\text{=L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\text{WL}\dfrac{\text{WR}\dfrac{\mathbb{R}\dfrac{*}{(x-y)^2 \le 0 \vdash x = y}}{(x-y)^2 \le 0 \vdash x = y, p(x)}}{(x-y)^2 \le 0, p(y) \vdash x = y, p(x)} \quad \text{WL}\dfrac{\text{=R}\dfrac{\text{id}\dfrac{}{p(y), x = y \vdash p(y)}}{p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y), x = y \vdash p(x)}}{\text{cut}\ \dfrac{(x-y)^2 \le 0, p(y) \vdash p(x)}{\wedge\text{L}\ \dfrac{(x-y)^2 \le 0 \wedge p(y) \vdash p(x)}{\to\text{R}\ \vdash (x-y)^2 \le 0 \wedge p(y) \to p(x)}}}$$

Creatively Cutting to Transform Questions

$$\text{=R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$\text{=L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{(x-y)^2 \le 0 \vdash x = y}\ \mathbb{R}}{(x-y)^2 \le 0 \vdash x = y, p(x)}\ \text{WR}}{(x-y)^2 \le 0, p(y) \vdash x = y, p(x)}\ \text{WL} \qquad \dfrac{\dfrac{\dfrac{*}{p(y), x = y \vdash p(y)}\ \text{id}}{p(y), x = y \vdash p(x)}\ \text{=R}}{(x-y)^2 \le 0, p(y), x = y \vdash p(x)}\ \text{WL}}{\dfrac{(x-y)^2 \le 0, p(y) \vdash p(x)}{\dfrac{(x-y)^2 \le 0 \land p(y) \vdash p(x)}{\vdash (x-y)^2 \le 0 \land p(y) \to p(x)}\ \to\text{R}}\ \land\text{L}}\ \text{cut}$$

Outline

Summary: Proof Rules of Sequent Calculus

$$\neg R \ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\neg L \ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge L \ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L \ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R \ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R \ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L \ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut} \ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot L \ \frac{}{\Gamma, \mathit{false} \vdash \Delta}$$

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} (y \notin \Gamma, \Delta, \forall x\, p(x)) \qquad \exists R \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} (\text{arbitrary term } e)$$

$$\forall L \ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} (\text{arbitrary term } e) \qquad \exists L \ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} (y \notin \Gamma, \Delta, \exists x\, p(x))$$

Summary: Proof Rules of Sequent Calculus

$$\neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot L\ \frac{}{\Gamma, \mathit{false} \vdash \Delta}$$

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} (y \notin \Gamma, \Delta, \forall x\, p(x)) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} (\text{arbitrary term } e)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} (\text{arbitrary term } e) \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} (y \notin \Gamma, \Delta, \exists x\, p(x))$$

$$\mathbb{R}\ \frac{}{\Gamma \vdash \Delta} \qquad \Big(\text{if } \bigwedge_{P \in \Gamma} P \rightarrow \bigvee_{Q \in \Delta} Q \text{ is valid in } \mathrm{FOL}_{\mathbb{R}}\Big)$$

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