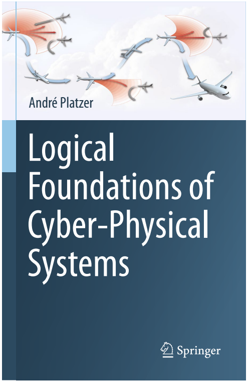
11: Differential Equations & Proofs

Logical Foundations of Cyber-Physical Systems



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Outline

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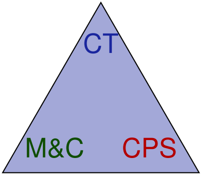
Learning Objectives

Differential Equations & Proofs

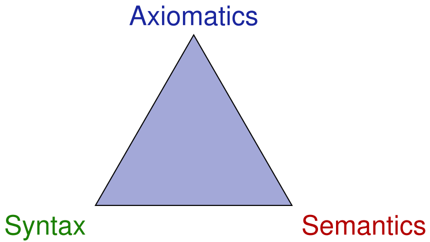
discrete vs. continuous analogy
rigorous reasoning about ODEs
beyond differential invariant terms
differential invariant formulas
cut principles for differential equations
axiomatization of ODEs
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous

operational CPS effects
state changes along ODE

Differential Facet of Logical Trinity



Syntax: defines the notation
What problems are we allowed to write down?

Semantics: what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics: internalizes semantic relations into universal syntactic transformations.
How does the semantics of $e \geq \tilde{e}$ relate to semantics of $e - \tilde{e} \geq 0$, syntactically? What about derivatives?

Outline

Differentials

Syntax $e ::= x \mid x' \mid c \mid e+k \mid e\cdot k \mid (e)'$

Semantics $\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$

Axioms

$$(e+k)' = (e)' + (k)'$$

$$(e\cdot k)' = (e)'\cdot k + e\cdot (k)'$$

$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

$$(x)' = x' \qquad \text{for variables } x \in \mathscr{V}$$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q \text{ for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z) \qquad \dots$$

Differential Substitution Lemmas $\rightsquigarrow$ Proofs

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r{>}0$*, then for all* $0{\leq}z{\leq}r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$\begin{aligned}
(e+k)' &= (e)' + (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(c())' &= 0 && \textit{for constants/numbers } c() \\
(x)' &= x' && \textit{for variables } x \in \mathcal{V}
\end{aligned}$$

Differential Substitution Lemmas $\rightsquigarrow$ Proofs

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r>0$*, then for all* $0 \leq z \leq r$*,* $FV(e) \subseteq \{x\}$*:*

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

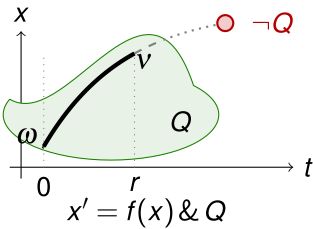
Lemma (Differential assignment) (Effect on Differentials)

$DE \quad [x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$+'$	$(e+k)' = (e)' + (k)'$
$\cdot'$	$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
c'	$(c())' = 0$
x'	$(x)' = x'$

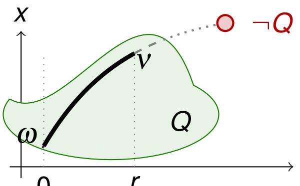
Differential Weakening



ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$
$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$
$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$$

Differential Weakening



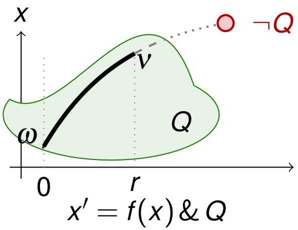
DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \wedge Q$$
$$\text{for some } \varphi : [0, r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$
$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$$

Differential equations cannot leave their domains.

Differential Weakening



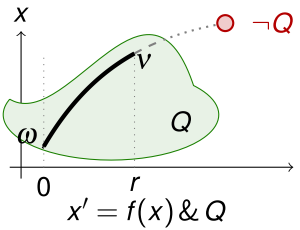
DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\text{DW}\;\frac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening

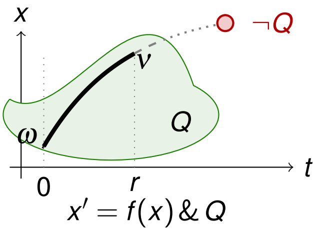


DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\dfrac{\dfrac{}{\vdash [x' = v, v' = -g\,\&\,x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\,\mathrm{G}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0]0 \leq x}\,\mathrm{DW}$$

No need to solve any ODEs to prove that bouncing ball is above ground.



DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\dfrac{\dfrac{\overline{\vdash x \geq 0 \rightarrow 0 \leq x}\;\mathbb{R}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\;\mathrm{G}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]0 \leq x}\;\mathrm{DW}$$

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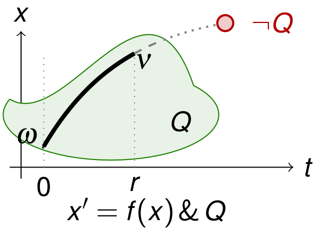

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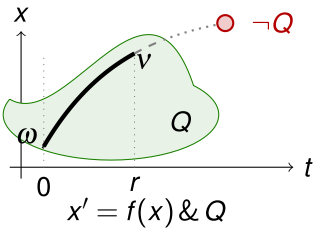
Example (Bouncing ball)

$$\dfrac{\dfrac{\dfrac{*}{\vdash x \geq 0 \rightarrow 0 \leq x}\ \mathbb{R}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\ \mathrm{G}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0]0 \leq x}\ \mathrm{DW}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening

$$\text{dW}\ \frac{}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$$



DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \to P)$

Example (Bouncing ball)

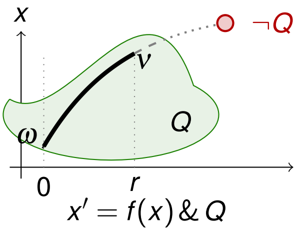
$$\dfrac{\dfrac{\dfrac{*}{\vdash x \geq 0 \to 0 \leq x}\ \mathbb{R}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0](x \geq 0 \to 0 \leq x)}\ \text{G}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0]\,0 \leq x}\ \text{DW}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening

Differential Weakening

$$\text{dW}\ \frac{Q \vdash P}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$$

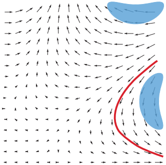


$$\text{DW}\ \ [x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow P)$$

Example (Bouncing ball)

$$\text{DW}\ \dfrac{\text{G}\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash x \geq 0 \rightarrow 0 \leq x}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0](x \geq 0 \rightarrow 0 \leq x)}}{\vdash [x' = v, v' = -g\,\&\,x \geq 0]0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.



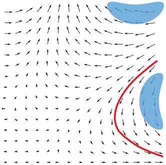
Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

DI $\big([x' = f(x)]\,e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)]\,(e)' = 0$

DE $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$

DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow P)$



Differential Invariant

$$\text{dI}\quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$$\text{DI}\quad \big([x' = f(x)\,\&\,Q]\,e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x)\,\&\,Q]\,(e)' = 0$$

$$\text{DE}\quad [x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$$

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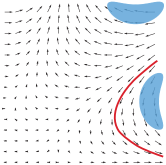


Differential Invariant

$$\text{dI}\ \frac{Q \vdash [x' := f(x)]\color{red}{(e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$$\text{DI}\ \big(\color{blue}{[x' = f(x)\,\&\,Q]\,e = 0} \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x)\,\&\,Q]\,\color{red}{(e)' = 0}$$

$$\text{DE}\ \color{blue}{[x' = f(x)\,\&\,Q]P} \leftrightarrow [x' = f(x)\,\&\,Q][\color{red}{x' := f(x)}]P$$

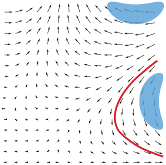
$$\text{DW}\ \color{blue}{[x' = f(x)\,\&\,Q]P} \leftrightarrow [x' = f(x)\,\&\,Q](\color{red}{Q} \rightarrow P)$$

Proof (dI is a derived rule).

$\square$

$$\text{DI}\ \frac{}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Differential Invariant Terms for Differential Equations



Differential Invariant

$$\text{dI}\ \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

DI $\big([x' = f(x)\,\&\,Q]\,e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x)\,\&\,Q]\,(e)' = 0$

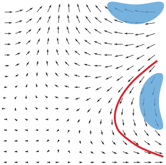
DE $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$

DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \to P)$

Proof (dI is a derived rule).

$$\text{DI}\ \dfrac{\text{DE}\ \dfrac{}{\vdash [x' = f(x)\,\&\,Q](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

□



Differential Invariant

$$\text{dI}\ \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

DI $\big([x' = f(x) \,\&\, Q]\, e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x) \,\&\, Q]\, (e)' = 0$

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

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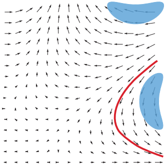
Proof (dI is a derived rule).

$$\dfrac{\dfrac{\dfrac{}{\vdash [x' = f(x) \,\&\, Q][x' := f(x)](e)' = 0}\ \text{DW}}{\vdash [x' = f(x) \,\&\, Q](e)' = 0}\ \text{DE}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}\ \text{DI}$$

$\square$



Differential Invariant

$$\text{dI}\ \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$$\text{DI}\ \big([x' = f(x)\,\&\,Q]\,e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x)\,\&\,Q]\,(e)' = 0$$

$$\text{DE}\ [x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$$

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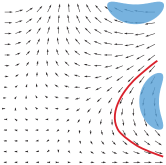
Proof (dI is a derived rule).

$$\text{DI}\ \cfrac{\text{DE}\ \cfrac{\text{DW}\ \cfrac{\text{G},{\rightarrow}\text{R}\ \cfrac{}{\vdash [x' = f(x)\,\&\,Q](Q \rightarrow [x' := f(x)](e)' = 0)}}{\vdash [x' = f(x)\,\&\,Q][x' := f(x)](e)' = 0}}{\vdash [x' = f(x)\,\&\,Q](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$\square$



Differential Invariant

$$\text{dI}\ \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$$\text{DI}\ \big([x' = f(x)\,\&\,Q]\,e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x)\,\&\,Q]\,(e)' = 0$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{Q \vdash [x' := f(x)](e)' = 0}{\vdash [x' = f(x)\,\&\,Q](Q \rightarrow [x' := f(x)](e)' = 0)}\ \text{G},\rightarrow\text{R}}{\vdash [x' = f(x)\,\&\,Q][x' := f(x)](e)' = 0}\ \text{DW}}{\vdash [x' = f(x)\,\&\,Q](e)' = 0}\ \text{DE}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}\ \text{DI}$$

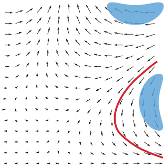
$$\text{G}\ \frac{P}{[\alpha]P}$$

□

Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0{\le}z{\le}r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$



Differential Invariant

$$\text{dI}\ \frac{}{e = k \vdash [x' = f(x)]e = k}$$

Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

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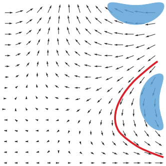
Differential Invariant

$$\mathrm{dI}\;\; \frac{\vdash [x' := f(x)]\,(e)' = (k)'}{e = k \vdash [x' = f(x)]\,e = k}$$



$$\mathrm{DI}\;\; \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Differential Invariant Equations

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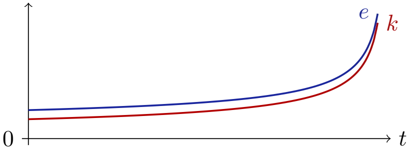
Differential Invariant

$$\text{dI}\;\; \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$



$$\text{DI}\;\; \big([x' = f(x)]\,e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Proof (= rate of change from = initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

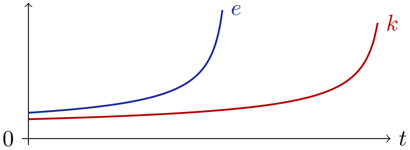
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \ \Rightarrow \ \forall 0 \le z \le r \ \ \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \ \frac{\vdash [x' := f(x)](e)' \ge (k)'}{e \ge k \vdash [x' = f(x)]e \ge k}$$



$$\text{DI} \ \left([x' = f(x)]\, e \ge k \leftrightarrow e \ge k\right) \leftarrow [x' = f(x)]\,(e)' \ge (k)'$$

Proof ($\ge$ rate of change from $\ge$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z) \qquad \square$$

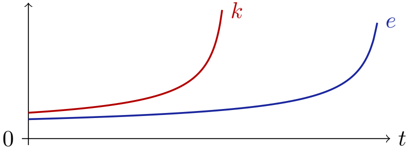
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \le (k)'}{e \le k \vdash [x' = f(x)]e \le k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \le k \leftrightarrow e \le k\big) \leftarrow [x' = f(x)]\,(e)' \le (k)'$$

Proof ($\le$ rate of change from $\le$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \le \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

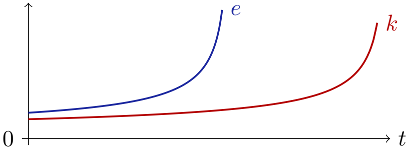
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\mathrm{dI}\;\; \frac{\vdash [x' := f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$



$$\mathrm{DI}\;\; \big([x' = f(x)]\,e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' > (k)'$$

Proof ($>$ rate of change from $>$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] > \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z)$$ $\square$

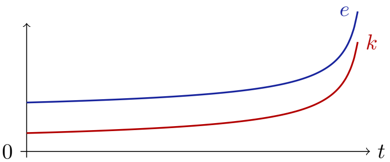
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \textit{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \;\; \frac{\vdash [x' := f(x)](e)' \geq (k)'}{e > k \vdash [x' = f(x)]e > k}$$



$$\text{DI} \;\; \big([x' = f(x)]\, e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' \geq (k)'$$

Proof ($\geq$ rate of change from $>$ initial value. Mean-value theorem).

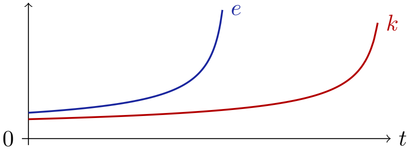
$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \geq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\mathrm{dI}\;\frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\mathrm{DI}\;\; \big([x' = f(x)]\,e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z)$$ □

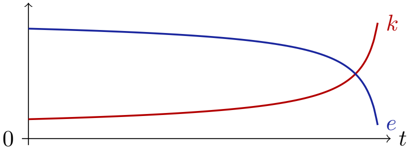
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \ \Rightarrow\ \forall 0{\le}z{\le}r\ \ \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI}\ \big([x' = f(x)]\,e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z)$$

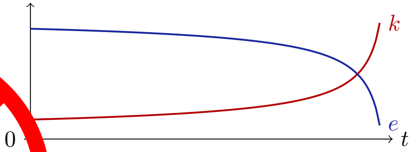
$\square$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \;\; \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \;\; \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)](e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

□

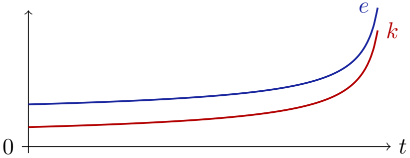
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \;\; \frac{\vdash [x' := f(x)](e)' = (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



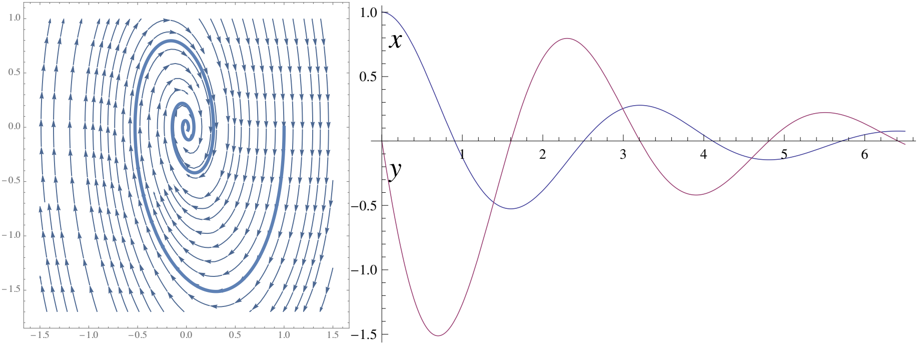
$$\text{DI} \;\; \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Proof ($=$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z) \qquad \square$$

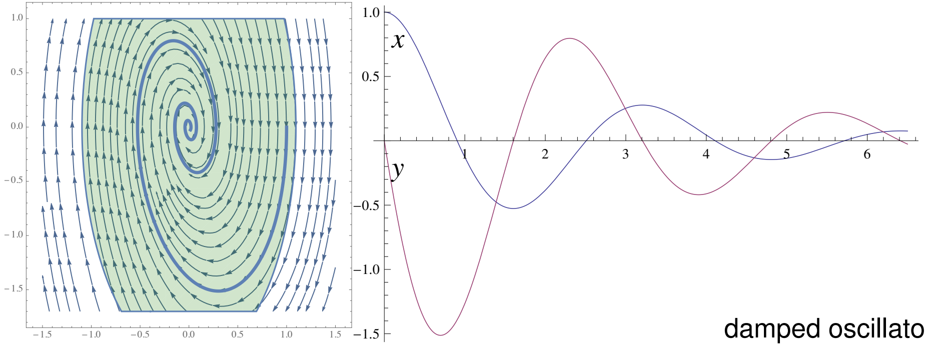
Example: Differential Invariant Inequalities

$$\frac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\ge 0 \land d\ge 0]\,\omega^2x^2+y^2\le c^2}$$



Example: Differential Invariant Inequalities: Oscillator

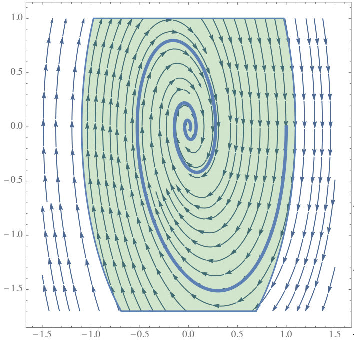
$$\frac{}{\omega^2 x^2 + y^2 \le c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \ge 0 \land d \ge 0]\, \omega^2 x^2 + y^2 \le c^2}$$

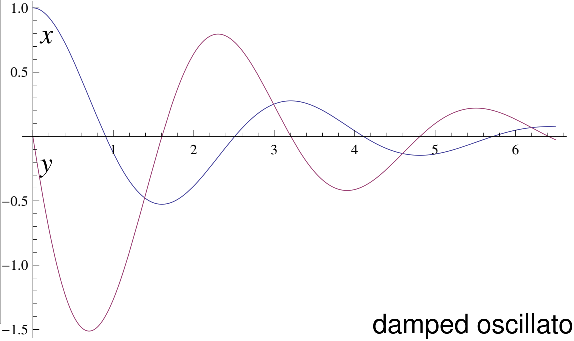


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\dfrac{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

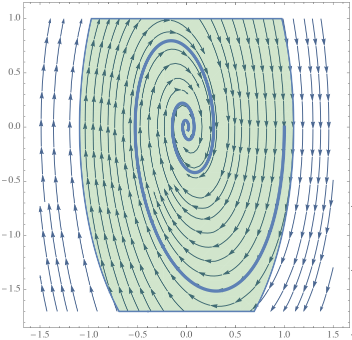


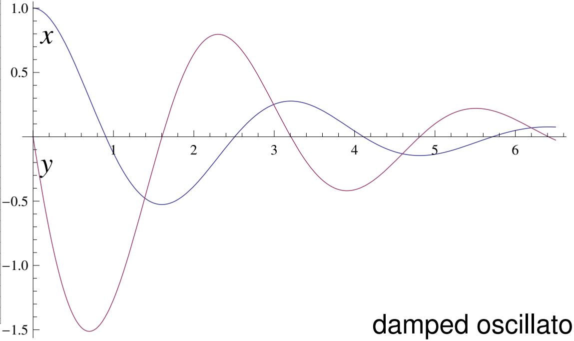


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\dfrac{\dfrac{\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}{\omega{\geq}0 \land d{\geq}0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}}{\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2 x^2 + y^2 \leq c^2}$$

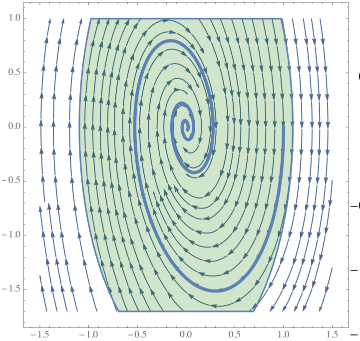


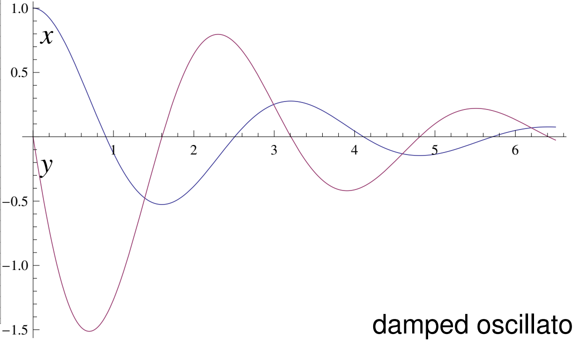


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$
\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \land d\geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}
$$

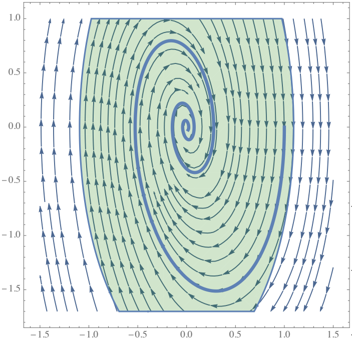


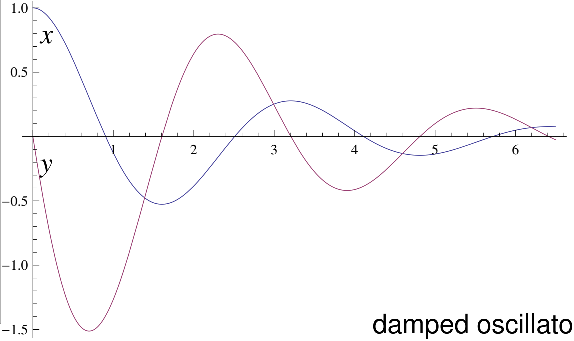


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

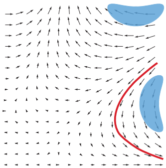
$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$





damped oscillator

Differential Invariant Conjunctions



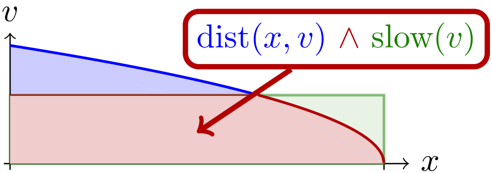
Differential Invariant

$$\text{dI}\quad \frac{}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$

Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$

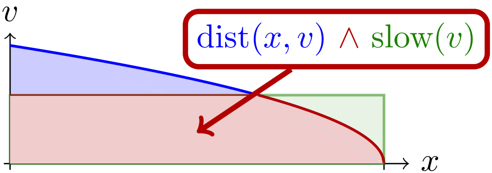


$$\text{DI}\ \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Proof (separately).

$$[]\land,\text{WL}\ \frac{\text{DI}\ \dfrac{\vdash [x' = f(x)](A)'}{A \vdash [x' = f(x)]A} \qquad \text{DI}\ \dfrac{\vdash [x' = f(x)](B)'}{B \vdash [x' = f(x)]B}}{A \land B \vdash [x' = f(x)](A \land B)}$$

$\square$

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

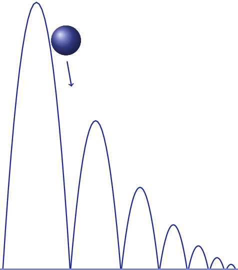
Quantum's Back for a Differential Invariant Proof

$$\dfrac{}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0](2gx=2gH-v^2 \land x\geq 0)}$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$[]\wedge \quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

$$[]\wedge \frac{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g\,\&\,x{\geq}0]2gx{=}2gH{-}v^2 \qquad \vdash [x''{=}{-}g\,\&\,x{\geq}0]x{\geq}0}{2gx{=}2gH{-}v^2 \vdash [x'' = -g\,\&\,x{\geq}0](2gx{=}2gH{-}v^2 \wedge x{\geq}0)}$$

No solutions but still a proof.

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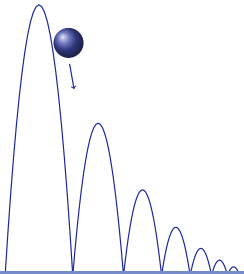

Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\overline{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0]2gx=2gH-v^2}\,\text{dI} \qquad \overline{\vdash [x''=-g\,\&\,x\geq 0]x\geq 0}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0](2gx=2gH-v^2 \land x\geq 0)}\,[]\land
$$

No solutions but still a proof.

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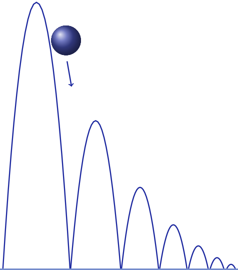

Quantum's Back for a Differential Invariant Proof

$$
{\scriptstyle []\wedge}\dfrac{{\scriptstyle \mathrm{dI}}\dfrac{{\scriptstyle [:=]}\dfrac{\dfrac{}{x\geq 0 \vdash 2g{\color{red}v} = -2v({\color{red}-g})}}{x\geq 0 \vdash [x':={\color{red}v}][v':={\color{red}-g}]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0]2gx=2gH-v^2} \qquad \dfrac{}{\vdash [x''=-g \,\&\, x\geq 0]x\geq 0}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0](2gx=2gH-v^2 \wedge x\geq 0)}
$$

No solutions but still a proof.

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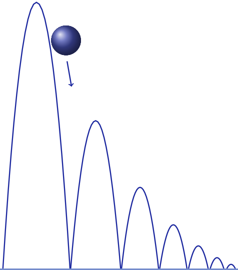
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$[]\land\;\dfrac{\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{x\geq0 \vdash 2gv = -2v(-g)}}{x\geq0\vdash [x':=v][v':=-g]2gx'=-2vv'}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0]2gx=2gH-v^2} \qquad \dfrac{}{\vdash [x''=-g\,\&\,x\geq0]x\geq0}}{2gx=2gH-v^2\vdash[x''=-g\,\&\,x\geq0](2gx=2gH-v^2\land x\geq0)}$$

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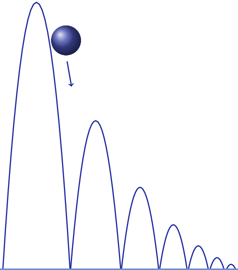

Quantum's Back for a Differential Invariant Proof

$$
{\scriptstyle []\wedge}\,\dfrac{{\scriptstyle \mathrm{dI}}\,\dfrac{{\scriptstyle [:=]}\,\dfrac{{\scriptstyle \mathbb{R}}\,\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}}{x\geq 0 \vdash [x' := v][v' := -g]2gx' = -2vv'}}{2gx = 2gH - v^2 \vdash [x'' = -g \,\&\, x\geq 0]2gx = 2gH - v^2} \qquad {\scriptstyle \mathrm{dW}}\,\dfrac{\overline{\color{red}{x\geq 0} \vdash x\geq 0}}{\vdash [x'' = -g \,\&\, \color{red}{x\geq 0}]x\geq 0}}{2gx = 2gH - v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx = 2gH - v^2 \wedge x\geq 0)}
$$

No solutions but still a proof.

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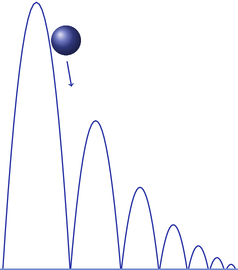


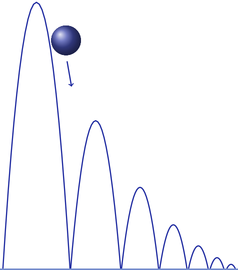
Quantum's Back for a Differential Invariant Proof

$$
[]\wedge\,\dfrac{
\text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0]2gx=2gH-v^2}
\quad
\text{dW}\,\dfrac{\text{id}\,\dfrac{*}{x\geq 0 \vdash x\geq 0}}{\vdash [x''=-g \,\&\, x\geq 0]x\geq 0}
}{2gx=2gH-v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx=2gH-v^2 \wedge x\geq 0)}
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

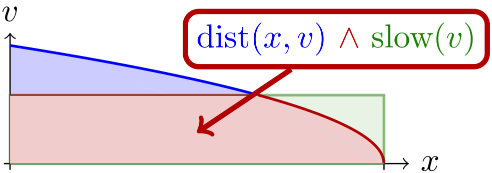
Independent proofs for independent questions.



Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$

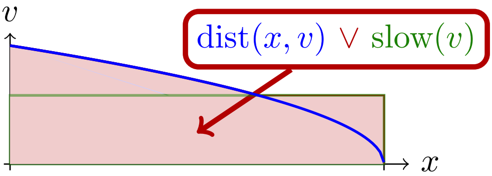


$$\text{DI}\ \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant Disjunctions

Differential Invariant

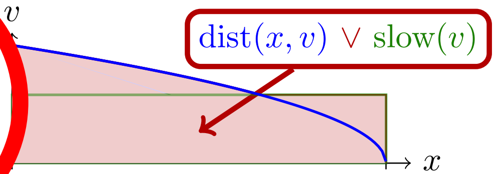
$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \lor (B)')$$

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

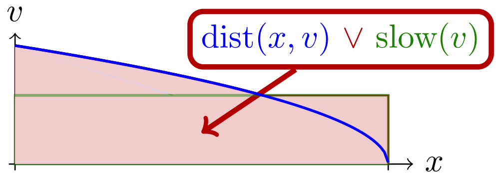


$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \lor (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

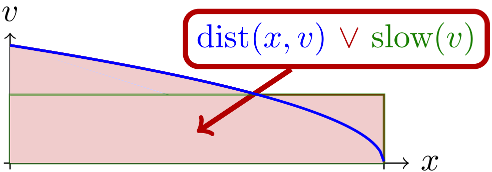


$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Proof (separately).

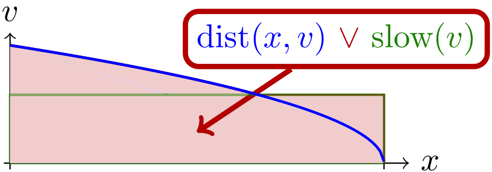
$$\vee\text{L}\ \dfrac{\text{MR}\ \dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \qquad \text{MR}\ \dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

$\square$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

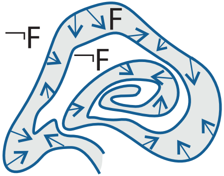
Proof (separately).

$$\text{∨L}\;\dfrac{\text{MR}\;\dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\;\dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \qquad \text{MR}\;\dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\;\dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

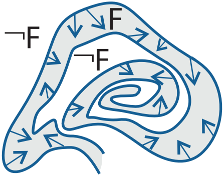
$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

□

Assuming Invariants



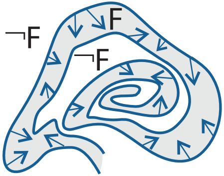
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



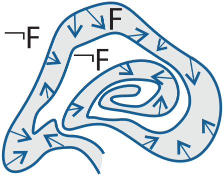
$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{loop} \quad \frac{F \vdash [\alpha]F}{F \vdash [\alpha^*]F}$$

Assuming Invariants



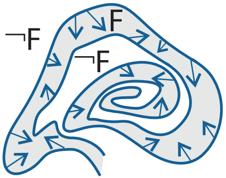
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

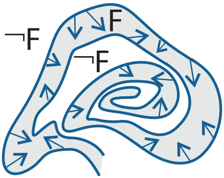


$$\frac{F \wedge Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

$$\frac{}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



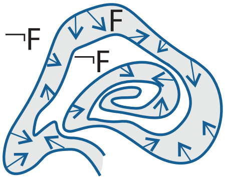


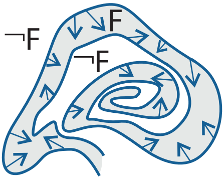
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

$$\frac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$





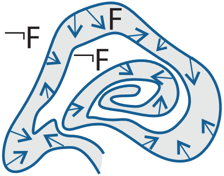
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$

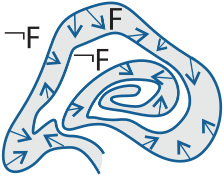
$$\frac{\color{red}{F} \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Example (Restrictions)

$$\cfrac{\cfrac{\cfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

Assuming Invariants

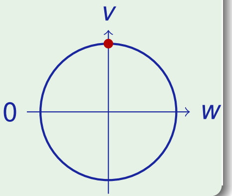




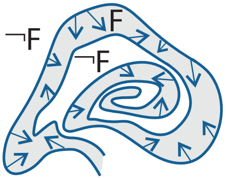
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F} \qquad \frac{\color{red}{F} \color{black}{\land Q \to [x' := f(x)](F)'}}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

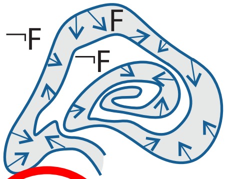
$$\dfrac{\dfrac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



Assuming Invariants



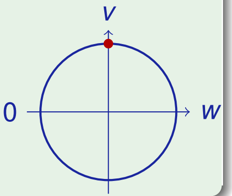
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



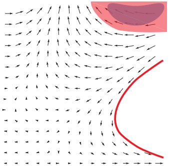
$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions are unsound!)

$$\dfrac{\dfrac{\dfrac{\text{(unsound)}}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



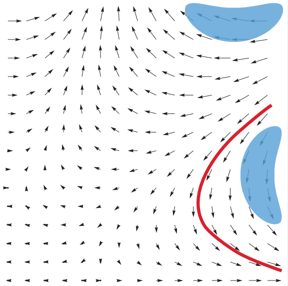
Outline



Differential Cut

$$\frac{}{F \vdash [x' = f(x)]F}$$

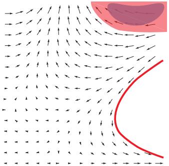
Differential Cut



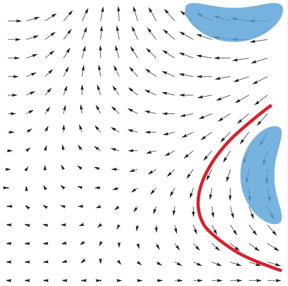
Differential Cuts

Differential Cut

$$\frac{F \vdash [x' = f(x)]C}{F \vdash [x' = f(x)]F}$$



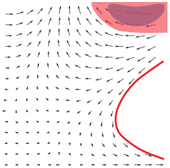
Differential Cut



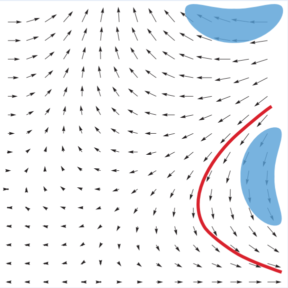
Differential Cuts

Differential Cut

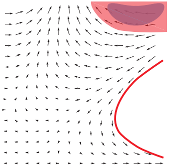
$$\frac{F \vdash [x' = f(x)]C \qquad F \vdash [x' = f(x) \,\&\, C]F}{F \vdash [x' = f(x)]F}$$



Differential Cut



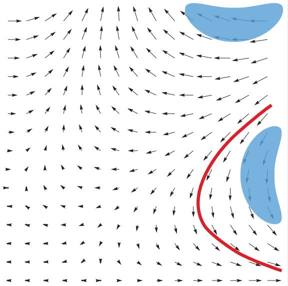
Differential Cuts



Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

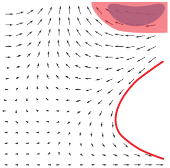
Differential Cut



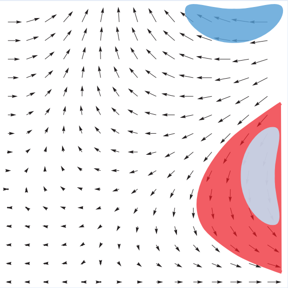
Differential Cuts

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



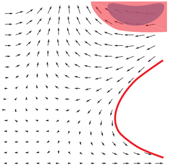
Differential Cut



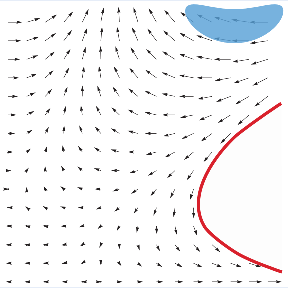
Differential Cuts

Differential Cut

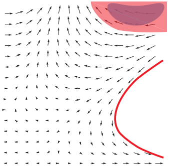
$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



Differential Cut



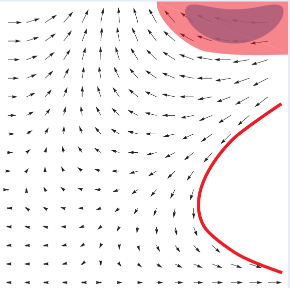
Differential Cuts



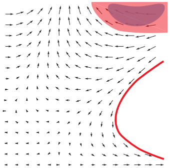
Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Differential Cut



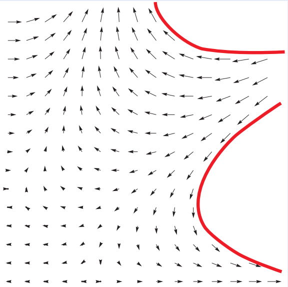
Differential Cuts



Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \wedge C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

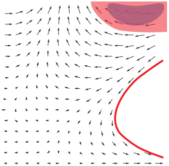
Differential Cut



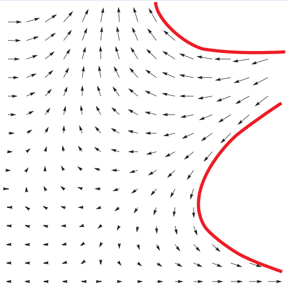
Differential Cuts

Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$



Differential Cut



Proof (Soundness).

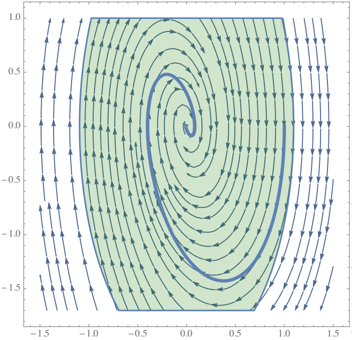
Let $\varphi \models x' = f(x) \land Q$ starting in $\omega \in [\![F]\!]$.
$\omega \in [\![[x' = f(x)\,\&\,Q]C]\!]$ by left premise.
Thus, $\varphi \models x' = f(x) \land Q \land C$.
Thus, $\varphi(r) \in [\![F]\!]$ by second premise. $\square$

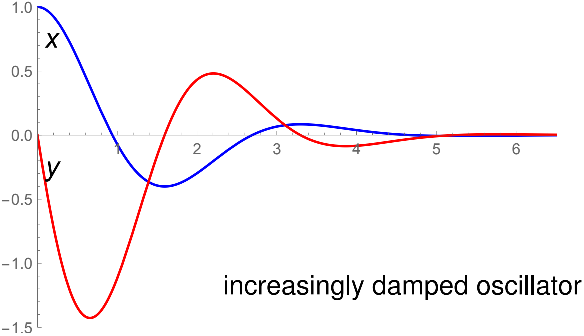
Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, {\color{red}d'=7} \,\&\, \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$





Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\dfrac{\text{dI}\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0\land d\ge 0]\,\omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\,\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0\land d\ge0]\,\omega^2x^2+y^2\le c^2}$$

$$\text{dC}\,\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,\omega^2x^2+y^2\le c^2}$$

$$\text{dI}\,\dfrac{}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,d\ge0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \dfrac{\text{dC}\ \dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge0\land d\ge0]\,\omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge0]\,\omega^2x^2+y^2\le c^2}$$

$$\text{[:=]}\ \dfrac{\text{dI}\ \dfrac{}{\omega\ge0 \vdash [d':=7]\,d'\ge0}}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge0]\,d\ge0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0\land d\ge 0]\,\omega^2x^2+y^2\le c^2}\text{dI}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0]\,\omega^2x^2+y^2\le c^2}\text{dC}$$

$$\dfrac{\dfrac{\dfrac{}{\omega\ge 0 \vdash 7\ge 0}\mathbb{R}}{\omega\ge 0 \vdash [d':=7]\,d'\ge 0}[:=]}{d\ge 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0]\,d\ge 0}\text{dI}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\;\dfrac{\text{dI}\;\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, {\color{red}d'=7} \,\&\, \omega\ge 0\wedge {\color{red}d\ge 0}]\,\omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, {\color{red}d'=7} \,\&\, \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\ge 0 \vdash 7\ge 0}}{\omega\ge 0 \vdash [d':=7]\,d'\ge 0}}{d\ge 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\,{\color{red}d\ge 0}}$$

ask

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\dfrac{\dfrac{}{\omega\geq 0\land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\,2\omega^2 x x' + 2yy' \leq 0}\,[:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2 x-2d\omega y, d'=7\,\&\,\omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2}\,\mathrm{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2 x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}\,\mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \vdash 7\geq 0}\,\mathbb{R}}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}\,[:=]}{d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}\,\mathrm{dI}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\mathrm{dC}\;\dfrac{\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{}{\omega\geq 0\wedge d\geq 0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\,2\omega^2 x x' + 2 y y' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}
$$

$$
\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}}{d\geq 0 \vdash [x'=y, y'=-\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}
$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\text{dC}\;\dfrac{\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}
$$

DC

$$
\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0\vdash 7\geq0}}{\omega\geq0\vdash [d':=7]\,d'\geq0}}{d\geq0\vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,d\geq0}
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increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \dfrac{\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}$$

init

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\omega\geq0 \vdash 7\geq0}}{\omega\geq0 \vdash [d':=7]\,d'\geq0}}{d\geq0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,d\geq0}$$

Differential Cut Example: Increasingly Damped Oscillator

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{\omega\geq 0\wedge d\geq 0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}\,\mathbb{R}}{\omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}\,[:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq c^2}\,\mathrm{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}\,\mathrm{dC}
$$

init

$$
\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \vdash 7\geq 0}\,\mathbb{R}}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}\,[:=]}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}\,\mathrm{dI}
$$

Could repeatedly diffcut in formulas to help the proof

Ex: Differential Cuts

$$\text{dC}\ \dfrac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

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$$\text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

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$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \cfrac{[:=]\ \cfrac{\mathbb{R}\ \cfrac{*}{\vdash 5y^4y^2 \geq 0}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]\color{red}{y^5 \geq 0}}$$

$$\dfrac{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}{x^3 \geq -1 \land {\color{red}y^5 \geq 0} \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}\;\text{dC}\quad(\text{dI above})$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\;\mathbb{R}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4y' \geq 0}\;[:=]}{{\color{red}y^5 \geq 0} \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}\;\text{dI}$$

Ex: Differential Cuts

$$\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash [x' := (x-2)^4 + y^5][y' := y^2] 3x^2 x' \geq 0}[:=]}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0] x^3 \geq -1 \quad \triangleright}\text{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2] x^3 \geq -1}\text{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4 y^2 \geq 0}\mathbb{R}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2] 5y^4 y' \geq 0}[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2] y^5 \geq 0}\text{dI}$$

$$
\begin{array}{ll}
 & \\ \hline
\mathbb{R} & y^5 \geq 0 \vdash 3x^2((x-2)^4 + y^5) \geq 0 \\ \hline
[:=] & y^5 \geq 0 \vdash [x' := (x-2)^4 + y^5][y' := y^2]3x^2 x' \geq 0 \\ \hline
\mathrm{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\ \hline
\mathrm{dC} & x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1
\end{array}
$$

$$
\begin{array}{ll}
 & * \\ \hline
\mathbb{R} & \vdash 5y^4 y^2 \geq 0 \\ \hline
[:=] & \vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4 y' \geq 0 \\ \hline
\mathrm{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
$$

Ex: Differential Cuts

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{y^5 \geq 0 \vdash 3x^2((x-2)^4 + y^5) \geq 0}\;\mathbb{R}}{y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]3x^2x' \geq 0}\;[:=]}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}\;\mathrm{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}\;\mathrm{dC}
$$

$$
\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\;\mathbb{R}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}\;[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]\color{red}{y^5 \geq 0}}\;\mathrm{dI}
$$

Outline

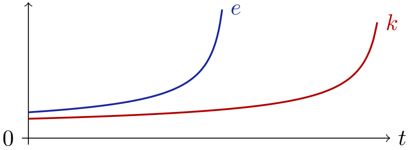
Soundness Proof: Differential Invariants

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \textit{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{DI} \quad \begin{array}{l} \big([x' = f(x)]e \ge 0 \leftrightarrow e \ge 0\big) \\ \leftarrow [x' = f(x)](e)' \ge 0 \end{array}$$



Proof ($\ge$ rate of change from $\ge$ initial value. Case $r = 0$ is easier.)

$h(t) \stackrel{\text{def}}{=} \varphi(t)[\![e]\!]$ is differentiable on $[0, r]$ if $r > 0$ by diff. lemma.

$\dfrac{\mathrm{d}h(t)}{\mathrm{d}t}(z) = \dfrac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \ge 0$ by lemma + assume for all z.

$h(r) - \underbrace{h(0)}_{\ge 0} = \underbrace{(r - 0)}_{>0} \underbrace{\dfrac{\mathrm{d}h(t)}{\mathrm{d}t}(\xi)}_{\ge 0} \ge 0$ by mean-value theorem for some ξ. $\square$

Outline

Summary: Differential Invariants for Differential Equations

Differential Weakening

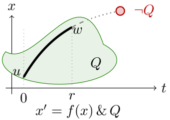
$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

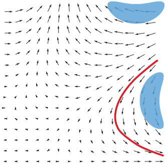
Differential Invariant

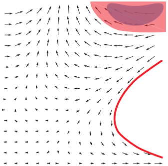
$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

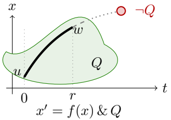
$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

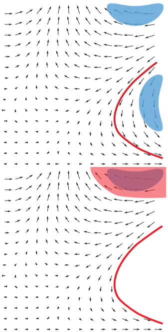






Summary: Differential Invariants for Differential Equations





Differential Weakening

$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [?Q]F\big) \leftarrow \big(Q \to [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3, doi:10.1007/978-3-319-63588-0.

André Platzer.
A complete uniform substitution calculus for differential dynamic logic.
J. Autom. Reas., 59(2):219–265, 2017.
doi:10.1007/s10817-016-9385-1.

André Platzer.
Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.
Springer, Heidelberg, 2010.
doi:10.1007/978-3-642-14509-4.

André Platzer.
Logics of dynamical systems.
In *LICS*, pages 13–24, Los Alamitos, 2012. IEEE.
doi:10.1109/LICS.2012.13.

André Platzer.
Differential-algebraic dynamic logic for differential-algebraic programs.
J. Log. Comput., 20(1):309–352, 2010.
doi:10.1093/logcom/exn070.

André Platzer.
The structure of differential invariants and differential cut elimination.
Log. Meth. Comput. Sci., 8(4:16):1–38, 2012.
doi:10.2168/LMCS-8(4:16)2012.

André Platzer.
A differential operator approach to equational differential invariants.
In Lennart Beringer and Amy Felty, editors, *ITP*, volume 7406 of *LNCS*, pages 28–48, Berlin, 2012. Springer.
doi:10.1007/978-3-642-32347-8_3.