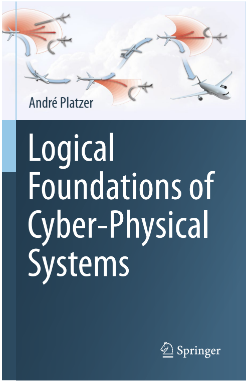
12: Ghosts & Differential Ghosts

Logical Foundations of Cyber-Physical Systems



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Logical Foundations of Cyber-Physical Systems

Springer

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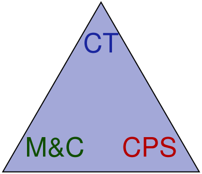
Outline

Outline

1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. A Gradual Introduction to Ghost Variables
   - Discrete Ghosts
   - Differential Ghosts of Time
   - Constructing Differential Ghosts
4. Differential Ghosts
   - Substitute Ghosts
   - Limit Velocity of an Aerodynamic Ball
5. Summary

Learning Objectives



extra dimensions for extra invariants

invent dark energy

CT

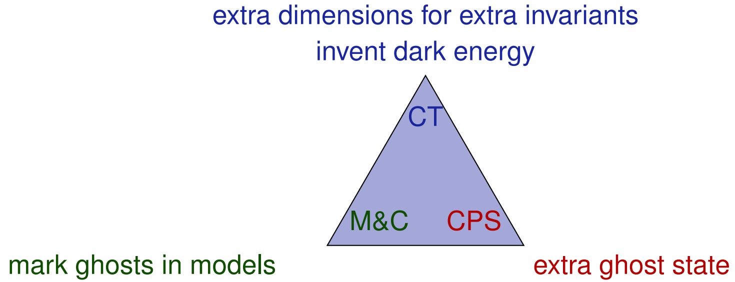
M&C

CPS

none: ghosts are for proofs!

extra ghost state

Learning Objectives



extra dimensions for extra invariants

invent dark energy

CT

M&C

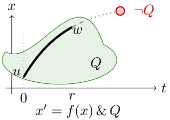
CPS

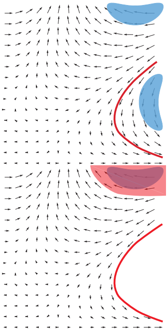
mark ghosts in models

extra ghost state

Outline

Differential Invariants for Differential Equations





Differential Weakening

$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

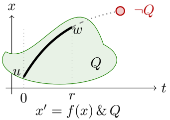
$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow F)$

DI $[x' = f(x) \,\&\, Q]F \leftarrow \big(Q \rightarrow F \land [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

Differential Invariants for Differential Equations

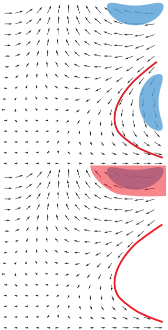


Differential Weakening

$$\frac{Q \vdash F}{P \vdash [x' = f(x)\,\&\,Q]F}$$

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$



Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

DW $[x' = f(x)\,\&\,Q]F \leftrightarrow [x' = f(x)\,\&\,Q](Q \to F)$

DI $[x' = f(x)\,\&\,Q]F \leftarrow \big(Q \to F \land [x' = f(x)\,\&\,Q](F)'\big)$

DC $\big([x' = f(x)\,\&\,Q]F \leftrightarrow [x' = f(x)\,\&\,Q \land C]F\big) \leftarrow [x' = f(x)\,\&\,Q]C$

DE $[x' = f(x)\,\&\,Q]F \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]F$

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5. Summary

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\quad (y \text{ new})$$

$$\rightarrow\!\text{R}\ \frac{}{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}$$

Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new})$$

$$\to\!\text{R}\ \dfrac{\text{iG}\ \dfrac{}{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1}}{\vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1}$$

Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta} \quad (y \text{ new})$$

discrete ghost c remembers function of old state

$$\dfrac{\dfrac{\dfrac{}{xy-1=0 \vdash [c := xy][x' = x, y' = -y]xy = 1}\,[:=]_=}{xy-1=0 \vdash [x' = x, y' = -y]xy = 1}\,\text{iG}}{\vdash xy-1=0 \rightarrow [x' = x, y' = -y]xy = 1}\,{\rightarrow}\text{R}$$

Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y\text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$$

$$\begin{array}{ll}
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
[:=]_= & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\to\text{R} & \overline{\vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1}
\end{array}$$

Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

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$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$
\begin{array}{ll}
\text{dI} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright} \\
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
[:=]_= & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}
$$

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Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_{=}\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{rc}
 & \\ \hline
[:=] & \vdash [x'{:=}x][y'{:=}-y]0 = x'y + xy' \\ \hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\ \hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\ \hline
[:=]_{=} & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\ \hline
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\ \hline
\rightarrow\text{R} & \vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1
\end{array}$$

Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new})$$

$p \leftrightarrow [y := e]p$ by $[:=]$

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{l|r}
\mathbb{R} & \\ \hline
 & \vdash 0 = xy + x(-y) \\ \hline
[:=] & \vdash [x' := x][y' := -y]0 = x'y + xy' \\ \hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \quad \triangleright \\ \hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\ \hline
[:=]_= & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\ \hline
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Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{lr}
 & * \\ \hline
\mathbb{R} & \vdash 0 = xy + x(-y) \\ \hline
[:=] & \vdash [x':=x][y':=-y]0 = x'y + xy' \\ \hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\ \hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\ \hline
[:=]_= & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\ \hline
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\ \hline
\to\text{R} & \vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1
\end{array}$$

Ask a ghost to remember some auxiliary state for the proof.

Solve by Differential Cuts and Differential Invariants

$$\dfrac{\dfrac{}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}\ \text{dC}}{v=0, a\geq 0, t=0 \vdash \qquad\qquad [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}\ \text{iG}$$

Solve by Differential Cuts and Differential Invariants

$$\text{iG}\;\dfrac{\text{dC}\;\dfrac{\text{[:=],dW}\;\overline{v=0,a\geq 0,t=0 \vdash [v_0:=v][x'=-vy,y'=vx,v'=a,t'=1 \,\&\, v=v_0+at]v\geq 0}}{v=0,a\geq 0,t=0 \vdash [v_0:=v][x'=-vy,y'=vx,v'=a,t'=1]v\geq 0}}{v=0,a\geq 0,t=0 \vdash \qquad\qquad [x'=-vy,y'=vx,v'=a,t'=1]v\geq 0}$$

Solve by Differential Cuts and Differential Invariants

$$\mathrm{iG}\;\dfrac{\mathrm{dC}\;\dfrac{[:=],\mathrm{dW}\;\dfrac{v_0=0, a\geq 0 \vdash t\geq 0 \land v=v_0+at \rightarrow v\geq 0}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1 \,\&\, v=v_0+at]v\geq 0}}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}}{v=0, a\geq 0, t=0 \vdash [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}$$

Differential Ghosts of Time

$$
\begin{array}{rl}
 & v_0 = 0, a \geq 0 \vdash t \geq 0 \land v = v_0 + at \to v \geq 0 \\ \hline
\text{[:=],dW} & v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \geq 0 \\ \hline
\text{dC} & v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0 \\ \hline
\text{iG} & v = 0, a \geq 0, t = 0 \vdash [x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0
\end{array}
$$

Why does the proof with ghost solutions need $t' = 1$ in the model?

Differential Ghosts of Time

$$\text{iG}\;\dfrac{\text{dC}\;\dfrac{[:=],\text{dW}\;\dfrac{v_0=0, a\geq 0 \vdash t\geq 0 \land v=v_0+at \rightarrow v\geq 0}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1 \,\&\, v=v_0+at]v\geq 0}}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}}{v=0, a\geq 0, t=0 \vdash \qquad [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}$$

Why does the proof with ghost solutions need $t'=1$ in the model?

Could we just add in $t'=1$ if we need it?

Differential Ghosts of Time

$$\dfrac{\dfrac{\dfrac{v_0 = 0, a \geq 0 \vdash t \geq 0 \land v = v_0 + at \rightarrow v \geq 0}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \geq 0}\,{\scriptstyle [:=],\mathrm{dW}}}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}\,{\scriptstyle \mathrm{dC}}}{v = 0, a \geq 0, t = 0 \vdash [x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}\,{\scriptstyle \mathrm{iG}}$$

Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

$$\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Ghosts of Time

$$
\begin{array}{rl}
 & \quad v_0 = 0, a \ge 0 \vdash t \ge 0 \land v = v_0 + at \to v \ge 0 \\ \hline
{\scriptstyle [:=],\mathrm{dW}} & v = 0, a \ge 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at] v \ge 0 \\ \hline
{\scriptstyle \mathrm{dC}} & v = 0, a \ge 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1] v \ge 0 \\ \hline
{\scriptstyle \mathrm{iG}} & v = 0, a \ge 0, t = 0 \vdash \qquad\qquad [x' = -vy, y' = vx, v' = a, t' = 1] v \ge 0
\end{array}
$$

Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

$$
\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}
$$

What could possibly go wrong?

Differential Ghosts of Time

$$\text{iG}\;\dfrac{\text{dC}\;\dfrac{[:=],\text{dW}\;\dfrac{v_0=0, a\geq 0 \vdash t\geq 0 \land v=v_0+at \rightarrow v\geq 0}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1 \,\&\, v=v_0+at]v\geq 0}}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}}{v=0, a\geq 0, t=0 \vdash [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}$$

Why does the proof with ghost solutions need $t'=1$ in the model?
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$$\frac{\Gamma \vdash [x'=f(x), t'=1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x'=f(x) \,\&\, Q]P, \Delta}$$

What could possibly go wrong?

- × Cannot add $t'=1$ to $x'=v, t'=2$

Differential Ghosts of Time

$$
\text{iG}\;\dfrac{\text{dC}\;\dfrac{[:=],\text{dW}\;\dfrac{v_0 = 0, a \geq 0 \vdash t \geq 0 \land v = v_0 + at \rightarrow v \geq 0}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \geq 0}}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}}{v = 0, a \geq 0, t = 0 \vdash \qquad [x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}
$$

Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

$$
\dfrac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}
$$

What could possibly go wrong?

- × Cannot add $t' = 1$ to $x' = v, t' = 2$
- × Cannot add $t' = 1$ to $x' = v, v' = t$

Differential Ghosts of Time

$$\text{iG}\;\dfrac{\text{dC}\;\dfrac{\text{[:=],dW}\;\dfrac{v_0 = 0, a \geq 0 \vdash t \geq 0 \land v = v_0 + at \to v \geq 0}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \geq 0}}{v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}}{v = 0, a \geq 0, t = 0 \vdash \qquad [x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0}$$

Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

$$\dfrac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

What could possibly go wrong?

- × Cannot add $t' = 1$ to $x' = v, t' = 2$
- × Cannot add $t' = 1$ to $x' = v, v' = t$
- ✓ Can add $t' = 1$ to $x' = v, v' = -g$

Differential Ghosts of Time

$$
\begin{array}{rl}
 & v_0=0, a\geq 0 \vdash t\geq 0 \land v=v_0+at \rightarrow v\geq 0 \\ \hline
\text{[:=],dW} & v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1 \,\&\, v=v_0+at]v\geq 0 \\ \hline
\text{dC} & v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0 \\ \hline
\text{iG} & v=0, a\geq 0, t=0 \vdash \qquad [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0
\end{array}
$$

Why does the proof with ghost solutions need $t'=1$ in the model?
Could we just add in $t'=1$ if we need it?

$$
\frac{\Gamma \vdash [x'=f(x), t'=1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x'=f(x) \,\&\, Q]P, \Delta}
$$

What could possibly go wrong?

- × Cannot add $t'=1$ to $x'=v, t'=2$
- × Cannot add $t'=1$ to $x'=v, v'=t$
- × Can add $t'=1$ to $x'=v, v'=-g$ unless e.g. postcondition P reads t

Differential Ghosts of Time

$$\cfrac{\cfrac{\cfrac{v_0=0, a\geq 0 \vdash t\geq 0 \land v=v_0+at \to v\geq 0}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1 \,\&\, v=v_0+at]v\geq 0}\text{[:=],dW}}{v=0, a\geq 0, t=0 \vdash [v_0:=v][x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}\text{dC}}{v=0, a\geq 0, t=0 \vdash \quad [x'=-vy, y'=vx, v'=a, t'=1]v\geq 0}\text{iG}$$

Why does the proof with ghost solutions need $t'=1$ in the model?
Could we just add in $t'=1$ if we need it?

$$\frac{\Gamma \vdash [x'=f(x), t'=1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x'=f(x) \,\&\, Q]P, \Delta} \quad (t \text{ fresh})$$

What could possibly go wrong?

- × Cannot add $t'=1$ to $x'=v, t'=2$
- × Cannot add $t'=1$ to $x'=v, v'=t$
- × Can add $t'=1$ to $x'=v, v'=-g$ unless e.g. postcondition P reads t

But this proof rule is too specific (for t only)

Differential Ghosts of Time

$$
\begin{array}{rl}
 & v_0 = 0, a \geq 0 \vdash t \geq 0 \land v = v_0 + at \to v \geq 0 \\ \hline
\text{[:=],dW} & v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \geq 0 \\ \hline
\text{dC} & v = 0, a \geq 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0 \\ \hline
\text{iG} & v = 0, a \geq 0, t = 0 \vdash \qquad [x' = -vy, y' = vx, v' = a, t' = 1]v \geq 0
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Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

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\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta} \quad (t \text{ fresh})
$$

Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x, y) \,\&\, Q]P$

Differential Ghosts of Time

$$\dfrac{\dfrac{\dfrac{v_0 = 0, a \ge 0 \vdash t \ge 0 \land v = v_0 + at \to v \ge 0}{v = 0, a \ge 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1 \,\&\, v = v_0 + at]v \ge 0}\,[:=],\mathrm{dW}}{v = 0, a \ge 0, t = 0 \vdash [v_0 := v][x' = -vy, y' = vx, v' = a, t' = 1]v \ge 0}\,\mathrm{dC}}{v = 0, a \ge 0, t = 0 \vdash [x' = -vy, y' = vx, v' = a, t' = 1]v \ge 0}\,\mathrm{iG}$$

Why does the proof with ghost solutions need $t' = 1$ in the model?
Could we just add in $t' = 1$ if we need it?

$$\dfrac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta} \quad (t \text{ fresh})$$

Get differential ghosts of time by axiom DG, even with initial $t = 0$:

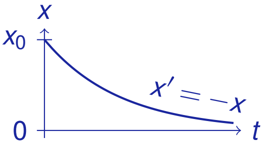
$$\dfrac{\dfrac{\Gamma, t = 0 \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash \exists t\, [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}\,\exists\mathrm{R}}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}\,\mathrm{DG}$$

Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x, y) \,\&\, Q]P$

Differential Ghosts for Exponentials

Example (Exponential decay)

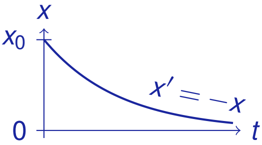
$$\text{dI}\ \frac{}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts for Exponentials

Example (Exponential decay)

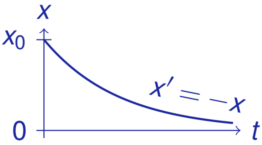
$$\text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [x' := -x] x' > 0}}{x > 0 \vdash [x' = -x] x > 0}$$



Differential Ghosts for Exponentials

Example (Exponential decay)

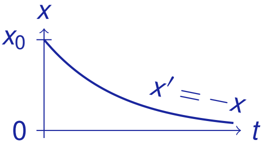
$$\text{dI}\;\cfrac{[:=]\;\cfrac{\mathbb{R}\;\cfrac{}{\vdash -x > 0}}{\vdash [x' := -x]x' > 0}}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts for Exponentials

Example (Cannot prove like this)

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash -x > 0}\;\mathbb{R}}{\vdash [x' := -x]x' > 0}\;[:=]}{x > 0 \vdash [x' = -x]x > 0}\;\text{dI}$$

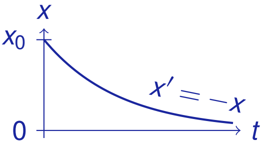


Differential Ghosts for Exponentials

Example (Cannot prove like this)

$$\cfrac{\cfrac{\cfrac{\text{not valid}}{\vdash -x > 0}\;\mathbb{R}}{\vdash [x' := -x]x' > 0}\;[:=]}{x > 0 \vdash [x' = -x]x > 0}\;\text{dI}$$

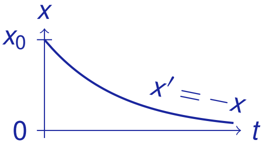
Matters get worse over time in this dynamics



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

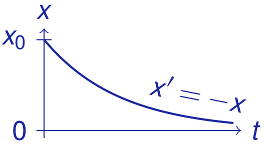
$$\text{DG}\ \frac{}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

$$\dfrac{\dfrac{}{x > 0 \vdash \exists y\,[x' = -x, {\color{red}y' = \tfrac{y}{2}}]x > 0}\,\exists\text{R,cut}}{x > 0 \vdash [x' = -x]x > 0}\,\text{DG}$$

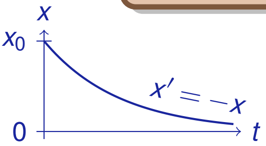


Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

$$\text{DG}\ \dfrac{\exists\text{R,cut}\ \dfrac{\text{MR}\ \dfrac{}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash [x' = -x]x > 0}$$

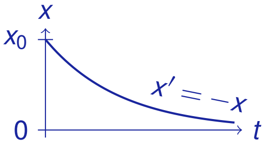
differential ghost: dream me up



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

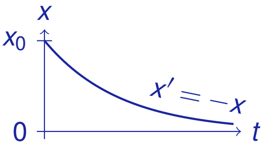
$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\overset{\mathbb{R}}{\overline{xy^2{=}1 \vdash x{>}0}} \qquad \overset{\text{dI}}{\overline{xy^2{=}1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

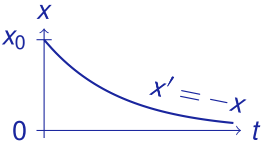
$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\;\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\;\dfrac{}{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

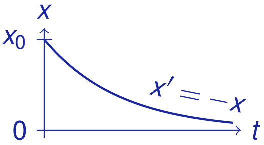
$$
\text{DG}\,\dfrac{\exists\text{R,cut}\,\dfrac{\text{MR}\,\dfrac{\mathbb{R}\,\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}
$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

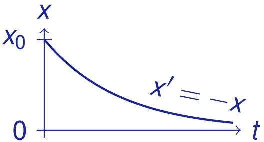
$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\;\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

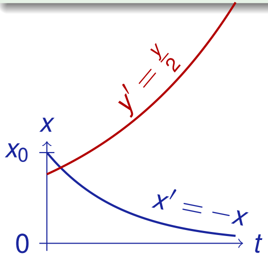
$$\text{DG}\,\dfrac{\exists\text{R,cut}\,\dfrac{\text{MR}\,\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\,\dfrac{[{:=}]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-}x][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof with counterweight ghost)

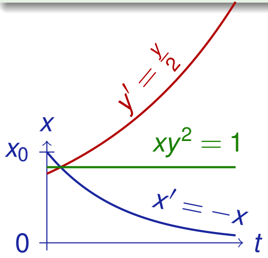
$$
\mathrm{DG}\;\dfrac{\exists\mathrm{R,cut}\;\dfrac{\mathrm{MR}\;\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \mathrm{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2+x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1\vdash[x'=-x,y'=\frac{y}{2}]x>0}}{x>0\vdash \exists y\,[x'=-x,{\color{red}y'=\frac{y}{2}}]x>0}}{x>0\vdash[x'=-x]x>0}
$$



Differential Ghosts: Proofs in Extra Dimensions

Example (Spooky proof with counterweight ghost)

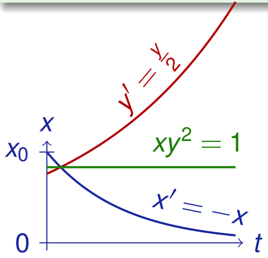
$$\text{DG}\,\dfrac{\exists\text{R,cut}\,\dfrac{\text{MR}\,\dfrac{\mathbb{R}\,\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof with counterweight ghost)

$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\;\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$

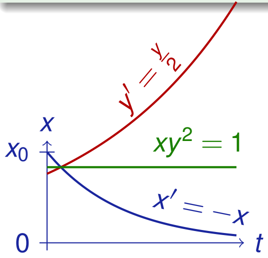


Creative proofs with differential ghosts prove what we otherwise couldn't!

Differential Ghosts: Proofs in Extra Dimensions

Example (Spooky proof with counterweight ghost)

$$
\text{DG}\;\dfrac{\text{∃R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\;\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}
$$



Creative proofs with differential ghosts prove what we otherwise couldn't!

Wait, are differential ghosts actually sound?

Outline

Nonexistent Differential Ghosts

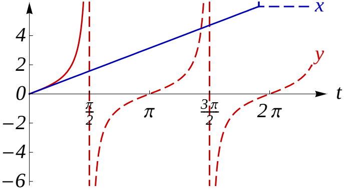
What could possibly go wrong?

$$\begin{array}{ll}
 & x{=}0, y{=}0 \vdash [x' = 1, y' = y^2 + 1]\, x \le 6 \\ \cline{2-2}
\exists\mathrm{R} & x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \le 6 \\ \cline{2-2}
\lightning & x = 0 \vdash [x' = 1]\, x \le 6
\end{array}$$

Nonexistent Differential Ghosts

What could possibly go wrong?

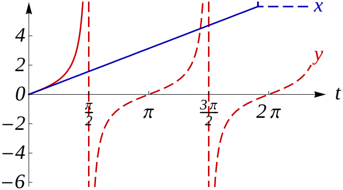
$$\text{↯}\ \cfrac{\exists\mathrm{R}\ \cfrac{x=0, y=0 \vdash [x' = 1, y' = y^2 + 1]\, x \le 6}{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \le 6}}{x = 0 \vdash [x' = 1]\, x \le 6}$$



Nonexistent Differential Ghosts

What could possibly go wrong? Explosive ghosts stop the world!

$$\text{↯}\ \dfrac{\exists\text{R}\ \dfrac{x{=}0, y{=}0 \vdash [x' = 1, y' = y^2 + 1]\, x \leq 6}{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \leq 6}}{x = 0 \vdash [x' = 1]\, x \leq 6}$$



Constructing Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

Constructing Differential Ghosts

Differential Ghost

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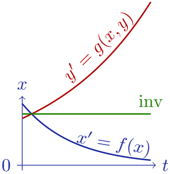
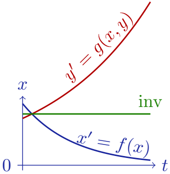


Constructing Differential Ghosts



Differential Ghost

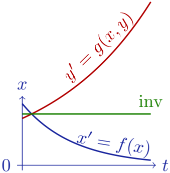
$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

if new $y' = g(x,y)$ has a global solution $y : [0,\infty) \to \mathbb{R}^n$

Differential Ghosts

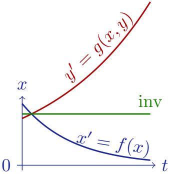
Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$



since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \& Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \& Q]P$$

Differential Ghost

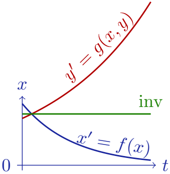
$$\text{dG}\quad \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \& Q]P, \Delta}{\Gamma \vdash [x' = f(x) \& Q]P, \Delta}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

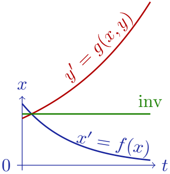


Differential Ghost

$$\text{dG} \quad \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x)\,\&\,Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x)\,\&\,Q]P$$

Differential Ghost

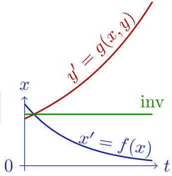
$$\text{dG}\quad \frac{\Gamma \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x)\,\&\,Q]P, \Delta}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$$

Differential Auxiliary

$$\text{dA}\quad \frac{\vdash F \leftrightarrow \exists y\,G \quad G \vdash [x' = f(x), y' = a(x)y + b(x)\,\&\,Q]G}{F \vdash [x' = f(x)\,\&\,Q]F}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG}\quad \frac{\Gamma \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

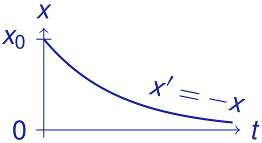
Differential Auxiliary

$$\text{dA}\quad \frac{\vdash F \leftrightarrow \exists y\, G \qquad G \vdash [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{DG}\,\dfrac{\text{MR}\,\dfrac{\dfrac{\exists y\, G \vdash F}{G \vdash F} \quad \exists\text{R,cut}\,\dfrac{F \vdash \exists y\, G \qquad G \vdash [x' = f(x), y' = a(x)y + b(x)]G}{F \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x)]G}}{F \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x)]F}}{F \vdash [x' = f(x)]F}$$

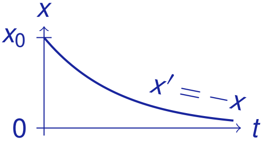
Substitute Ghosts

dA $$\frac{}{x > 0 \vdash [x' = -x]x > 0}$$



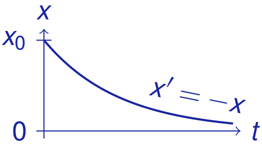
Substitute Ghosts

$$\text{dA}\;\cfrac{\mathbb{R}\;\cfrac{}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\cfrac{}{xy^2=1 \vdash [x' = -x, y' = \text{☁}]xy^2 = 1}}{x > 0 \vdash [x' = -x]x > 0}$$



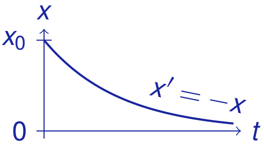
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{}{xy^2=1 \vdash [x' = -x, y' = \text{☁}]xy^2 = 1}}{x > 0 \vdash [x' = -x]x > 0}$$



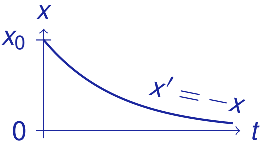
Substitute Ghosts

$$\text{dA}\;\cfrac{\mathbb{R}\;\cfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\cfrac{[:=]\;\cfrac{}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



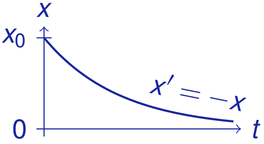
Substitute Ghosts

$$
\dfrac{\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1}\,\mathbb{R} \qquad \dfrac{\dfrac{\dfrac{}{\vdash -x y^2+2xy\,\text{☁}=0}}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}\,[:=]}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}\,\text{dI}}{x>0 \vdash [x'=-x]x>0}\,\text{dA}
$$



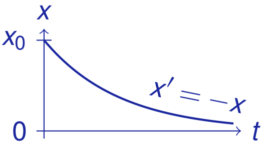
Substitute Ghosts

$$\mathrm{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \mathrm{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{\text{could prove if } \text{☁} = \frac{y}{2}}{\vdash -xy^2+2xy\text{☁}=0}}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



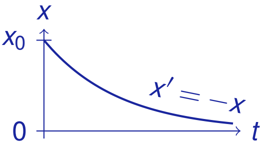
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{\text{could prove if } \text{☁} = \frac{y}{2}}{\vdash -xy^2+2xy\,\text{☁}=0}}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



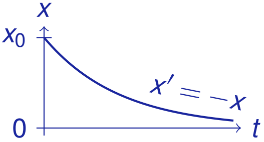
Substitute Ghosts

$$\mathrm{dA}\;\cfrac{\mathbb{R}\;\cfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \mathrm{dI}\;\cfrac{[:=]\;\cfrac{\cfrac{\text{could prove if } \frac{y}{2} \quad = \frac{y}{2} \text{ proved!}}{\vdash -xy^2+2xy\frac{y}{2} \quad =0}}{\vdash [x':=-x][y':=\frac{y}{2} \quad]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \quad]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



Substitute Ghosts

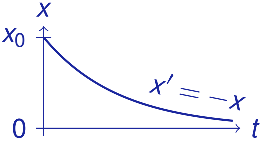
$$
\begin{prooftree}
\AxiomC{$*$}
\LeftLabel{$\mathbb{R}$}
\UnaryInfC{$\vdash x>0 \leftrightarrow \exists y\, xy^2=1$}
\AxiomC{could prove if $\frac{y}{2} \qquad = \frac{y}{2}$}
\UnaryInfC{$\vdash -xy^2 + 2xy\frac{y}{2} \qquad = 0$}
\LeftLabel{$[:=]$}
\UnaryInfC{$\vdash [x':=-x][y':=\frac{y}{2} \qquad]x'y^2+x2yy'=0$}
\LeftLabel{dI}
\UnaryInfC{$xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \qquad]xy^2=1$}
\LeftLabel{dA}
\BinaryInfC{$x>0 \vdash [x'=-x]x>0$}
\end{prooftree}
$$



This is a recipe for brewing suitable differential ghosts!

Substitute Ghosts

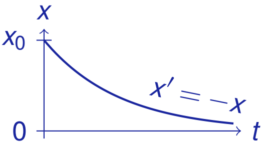
$$\mathrm{dA}\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \mathrm{dI}\,\dfrac{[:=]\,\dfrac{\dfrac{\text{could prove if } {\color{red}j(y)} = \frac{y}{2} \text{ proved!}}{\vdash -xy^2+2xy{\color{red}j(y)} = 0}}{\vdash [x':=-x][y':={\color{red}j(y)}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'={\color{red}j(y)}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



Function symbol $j(y)$ can play the role of a substitute ghost

Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{\text{could prove if } \frac{y}{2} \quad = \frac{y}{2}}{\vdash -xy^2+2xy\frac{y}{2} \quad = 0}}{\vdash [x':=-x][y':=\frac{y}{2} \quad]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \quad]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$

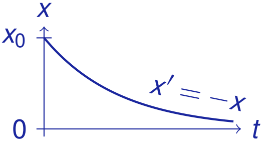


Function symbol $j(y)$ can be substituted uniformly

▸ Chapter 18

Substitute Ghosts

$$\text{dA}\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\dfrac{\text{could prove if } j(y) = \frac{y}{2}}{\vdash -xy^2+2xyj(y)=0}}{\vdash [x':=-x][y':=j(y)]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=j(y)]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$

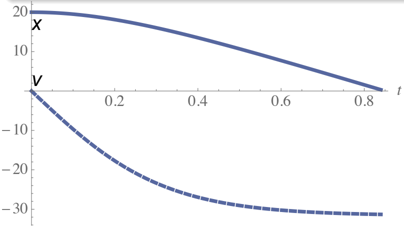


Function symbol $j(y)$ needs to be instantiated linearly in y

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

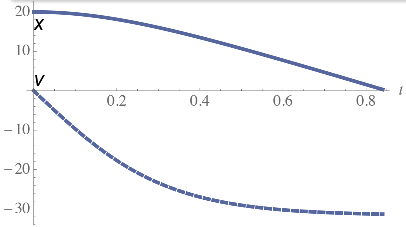
$$g>0 \land r>0 \qquad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \quad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



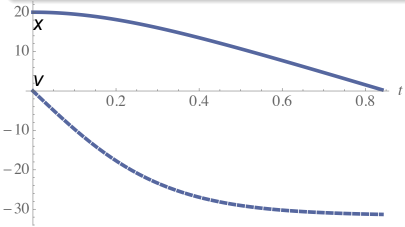
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \qquad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



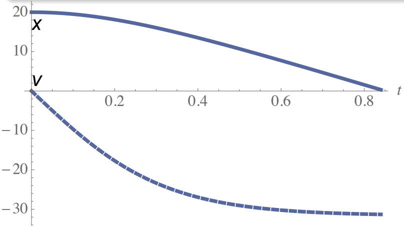
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

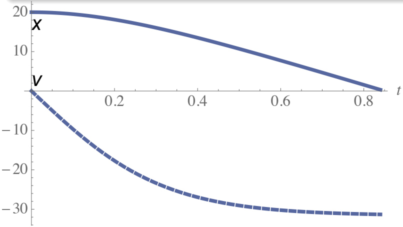
$v' = 0$ iff $-g+rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\text{dA}\ \frac{}{v>-\sqrt{g/r}\vdash [x'=v,v'=-g+rv^2]\,v>-\sqrt{g/r}}$$

Proposition (Aerodynamic velocity limits)

$$g>0\land r>0\land v>-\sqrt{\frac{g}{r}}\rightarrow [x'=v,v'=-g+rv^2\,\&\,x\geq 0\land v\leq 0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

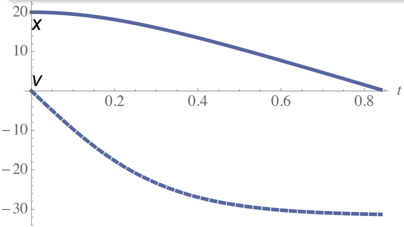
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\text{dA}\ \dfrac{\text{dI}\ \dfrac{}{y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, {\color{red}y'=j(x,v,y)}]y^2(v+\sqrt{g/r})=1} \quad \triangleright}{v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\, v>-\sqrt{g/r}}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

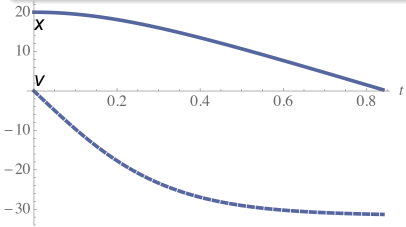
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\dfrac{\dfrac{\dfrac{}{\vdash [x':=v][v':=-g+rv^2][y':=j(x,v,y)]2yy'(v+\sqrt{g/r})+y^2v'=0}\,[:=]}{y^2(v+\sqrt{g/r})=1 \vdash [x'=v,v'=-g+rv^2,y'=j(x,v,y)]y^2(v+\sqrt{g/r})=1 \qquad \triangleright}\,\mathrm{dI}}{v>-\sqrt{g/r} \vdash [x'=v,v'=-g+rv^2]\,v>-\sqrt{g/r}}\,\mathrm{dA}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v,v'=-g+rv^2\,\&\,x\geq 0 \land v\leq 0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

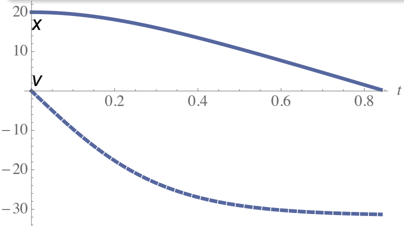
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\begin{array}{ll}
\hline
 & \vdash 2y(\color{red}{j(x,v,y)})(v+\sqrt{g/r})+y^2(-g+rv^2)=0 \\
\hline
[:=] & \vdash [x':=v][v':=-g+rv^2][\color{red}{y'}:=\color{red}{j(x,v,y)}]2y\color{red}{y'}(v+\sqrt{g/r})+y^2v'=0 \\
\hline
\mathrm{dI} & y^2(v+\sqrt{g/r})=1 \vdash [x'=v,v'=-g+rv^2,\color{red}{y'=j(x,v,y)}]y^2(v+\sqrt{g/r})=1 \quad \triangleright \\
\hline
\mathrm{dA} & v>-\sqrt{g/r} \vdash [x'=v,v'=-g+rv^2]\,v>-\sqrt{g/r}
\end{array}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v,v'=-g+rv^2\,\&\,x\geq0 \land v\leq0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

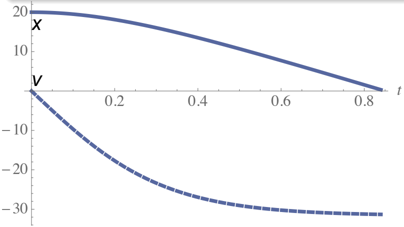
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$
\begin{array}{lrl}
\hline
\mathbb{R} & & \vdash -ry^2(v^2 - g/r) + y^2(-g + rv^2) = 0 \\
\hline
 & & \vdash 2y(-r/2(v-\sqrt{g/r})y)(v+\sqrt{g/r}) + y^2(-g+rv^2) = 0 \\
\hline
[:=] & & \vdash [x':=v][v':=-g+rv^2][y':=-r/2(v-\sqrt{g/r})y]2yy'(v+\sqrt{g/r})+y^2v'=0 \\
\hline
\mathrm{dI} & y^2(v+\sqrt{g/r})=1 & \vdash [x'=v, v'=-g+rv^2, y'=-r/2(v-\sqrt{g/r})y]y^2(v+\sqrt{g/r})=1 \quad \triangleright \\
\hline
\mathrm{dA} & v>-\sqrt{g/r} & \vdash [x'=v, v'=-g+rv^2]\, v > -\sqrt{g/r}
\end{array}
$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \to [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

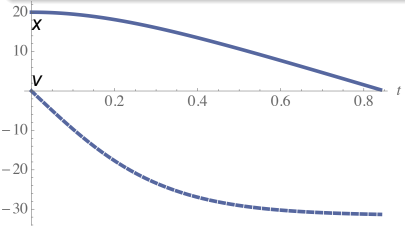
$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\begin{array}{lrl}
 & & * \\ \hline
\mathbb{R} & & \vdash -ry^2(v^2 - {}^{g}/_{r}) + y^2(-g + rv^2) = 0 \\ \hline
 & & \vdash 2y(-{}^{r}/_{2}(v - \sqrt{{}^{g}/_{r}})y)(v + \sqrt{{}^{g}/_{r}}) + y^2(-g + rv^2) = 0 \\ \hline
[:=] & & \vdash [x' := v][v' := -g + rv^2][y' := -{}^{r}/_{2}(v - \sqrt{{}^{g}/_{r}})y]2yy'(v + \sqrt{{}^{g}/_{r}}) + y^2 v' = 0 \\ \hline
\mathrm{dI} & y^2(v + \sqrt{{}^{g}/_{r}}) = 1 & \vdash [x' = v, v' = -g + rv^2, y' = -{}^{r}/_{2}(v - \sqrt{{}^{g}/_{r}})y]y^2(v + \sqrt{{}^{g}/_{r}}) = 1 \quad \triangleright \\ \hline
\mathrm{dA} & v > -\sqrt{{}^{g}/_{r}} & \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{{}^{g}/_{r}}
\end{array}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

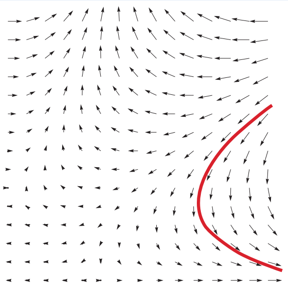
$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

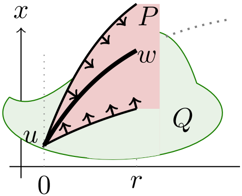
Outline

1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. A Gradual Introduction to Ghost Variables
   - Discrete Ghosts
   - Differential Ghosts of Time
   - Constructing Differential Ghosts
4. Differential Ghosts
   - Substitute Ghosts
   - Limit Velocity of an Aerodynamic Ball
5. Summary

Differential Invariants for Differential Equations

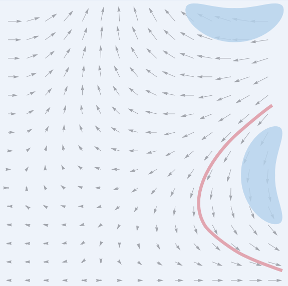
Differential Invariant

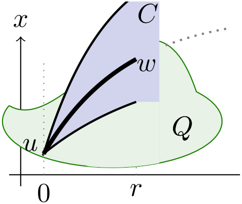




$$x' = f(x) \,\&\, Q$$

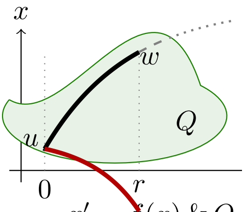
Differential Cut





$$x' = f(x) \,\&\, Q$$

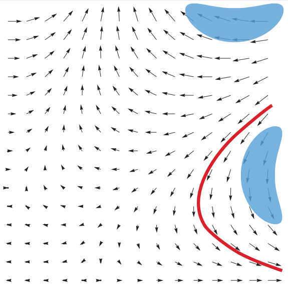
Differential Ghost

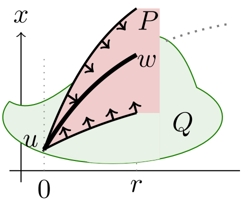


$$x' = f(x) \,\&\, Q$$

Differential Invariants for Differential Equations

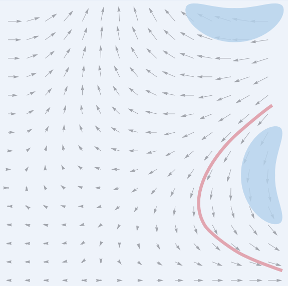
Differential Invariant

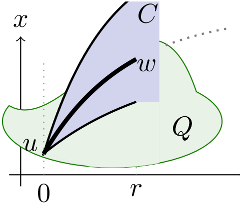




$$x' = f(x) \,\&\, Q$$

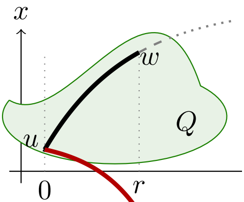
Differential Cut





$$x' = f(x) \,\&\, Q$$

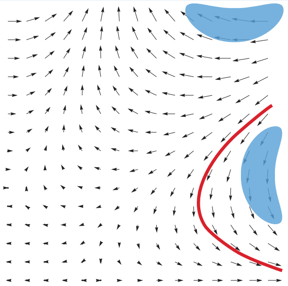
Differential Ghost

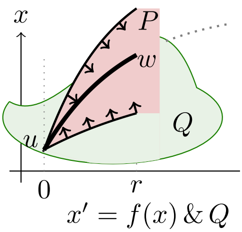


$$x' = f(x) \,\&\, Q$$

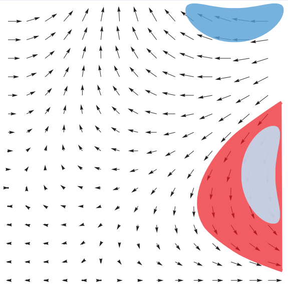
Differential Invariants for Differential Equations

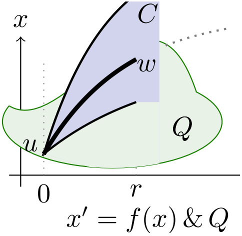
Differential Invariant



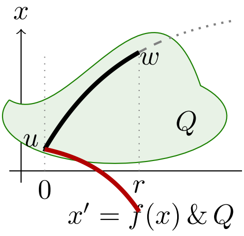


Differential Cut



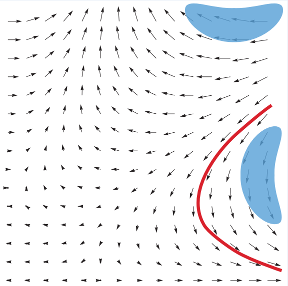


Differential Ghost

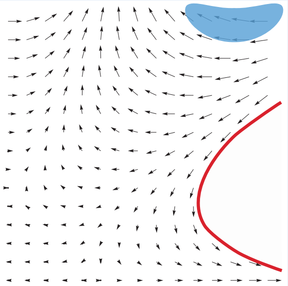


Differential Invariants for Differential Equations

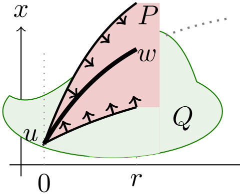
Differential Invariant



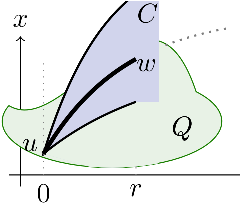
Differential Cut



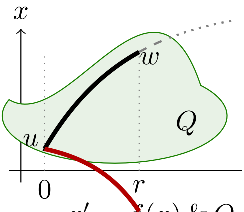
Differential Ghost



$x' = f(x) \,\&\, Q$



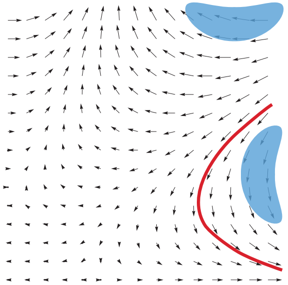
$x' = f(x) \,\&\, Q$

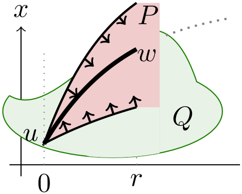


$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

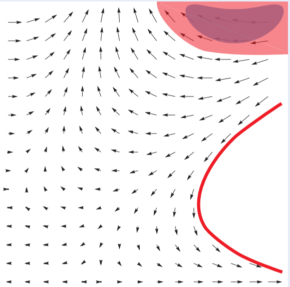
Differential Invariant

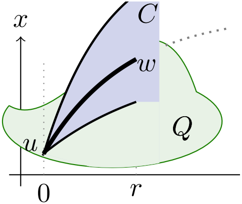




$x' = f(x) \,\&\, Q$

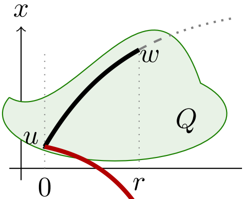
Differential Cut





$x' = f(x) \,\&\, Q$

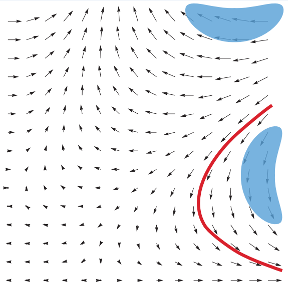
Differential Ghost

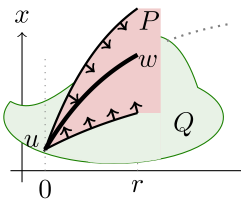


$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

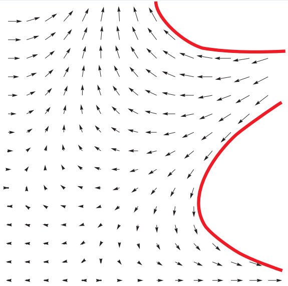
Differential Invariant

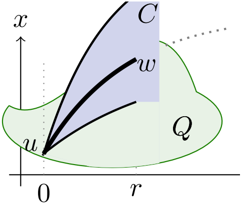




$x' = f(x) \,\&\, Q$

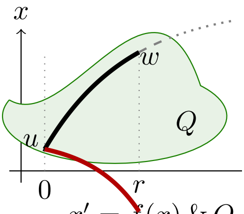
Differential Cut





$x' = f(x) \,\&\, Q$

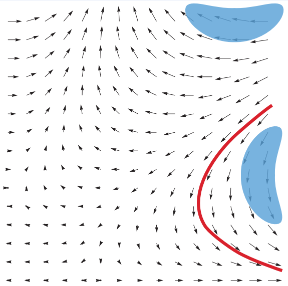
Differential Ghost

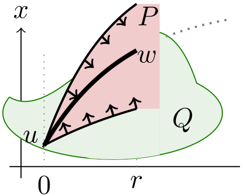


$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

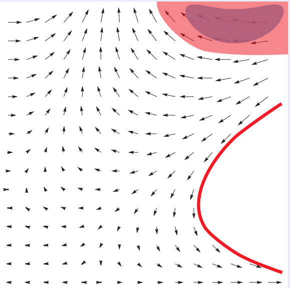
Differential Invariant

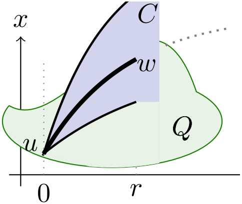




$$x' = f(x) \,\&\, Q$$

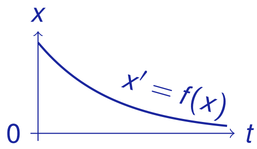
Differential Cut

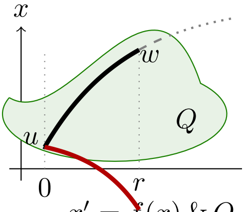




$$x' = f(x) \,\&\, Q$$

Differential Ghost

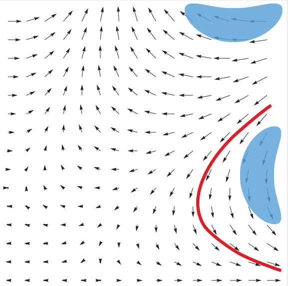


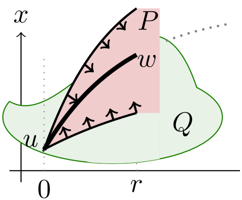


$$x' = f(x) \,\&\, Q$$

Differential Invariants for Differential Equations

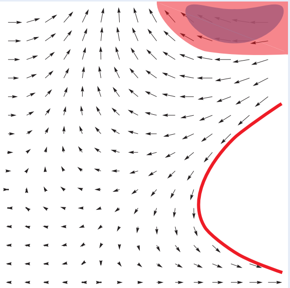
Differential Invariant

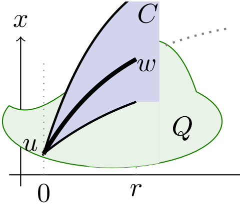




$x' = f(x) \,\&\, Q$

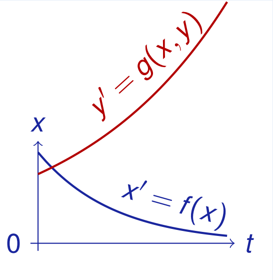
Differential Cut

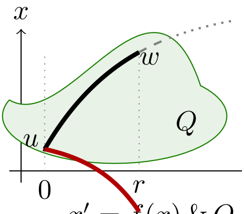




$x' = f(x) \,\&\, Q$

Differential Ghost

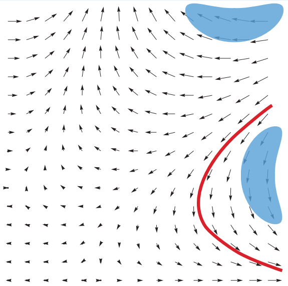




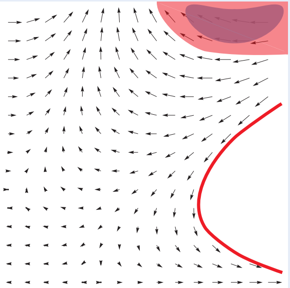
$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

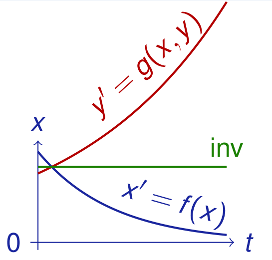
Differential Invariant

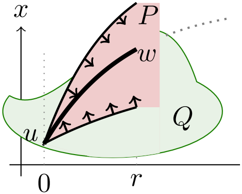


Differential Cut

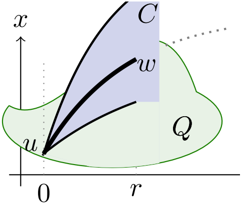


Differential Ghost

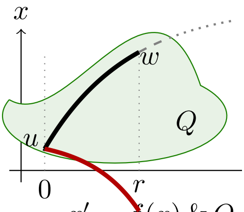




$$x' = f(x) \,\&\, Q$$



$$x' = f(x) \,\&\, Q$$

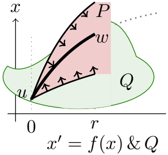


$$x' = f(x) \,\&\, Q$$

Differential Invariants for Differential Equations

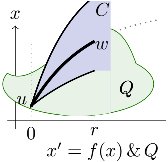
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](P)'}{P \vdash [x' = f(x) \,\&\, Q]P}$$



Differential Cut

$$\frac{P \vdash [x' = f(x) \,\&\, Q]C \quad P \vdash [x' = f(x) \,\&\, Q \wedge C]P}{P \vdash [x' = f(x) \,\&\, Q]P}$$



Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \vdash [x' = f(x) \,\&\, Q]P}$$

if new $y' = g(x,y)$ has long enough solution