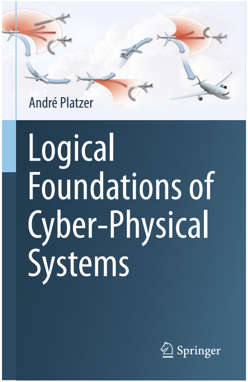
14: Hybrid Systems & Games

Logical Foundations of Cyber-Physical Systems



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Logical Foundations of Cyber-Physical Systems

Springer

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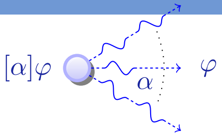
Computer Science Department

Dynamic Logics for Dynamical Systems

differential dynamic logic

dL = DL + HP



Logical Foundations of Cyber-physical Systems

discrete

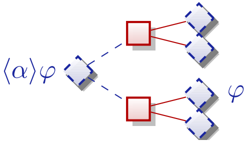
continuous

nondeterministic

…

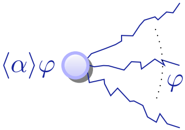
differential game logic

dGL = GL + HG



stochastic differential DL

$\mathrm{Sd}\mathscr{L} = \mathrm{DL} + \mathrm{SHP}$



quantified differential DL

$\mathrm{Qd}\mathscr{L} = \mathrm{FOL} + \mathrm{DL} + \mathrm{QHP}$

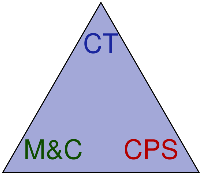
Outline

Outline

Learning Objectives

Hybrid Systems & Games

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
best/worst-case analysis
models of alternating computation



adversarial dynamics
conflicting actions
multi-agent systems
angelic/demonic choice

multi-agent state change
CPS semantics
reflections on choices

Outline

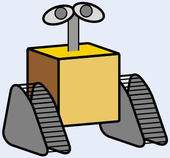
1. Learning Objectives
2. Motivation
3. A Gradual Introduction to Hybrid Games
    - Choices & Nondeterminism
    - Control & Dual Control
    - Demon's Derived Controls
4. Differential Game Logic
    - Syntax of Hybrid Games
    - Syntax of Differential Game Logic Formulas
    - Examples
    - Push-around Cart
    - Robot Dance
    - Example: Robot Soccer
5. An Informal Operational Game Tree Semantics
6. Summary

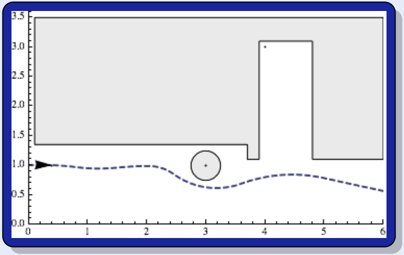
CPS Analysis: Robot Control

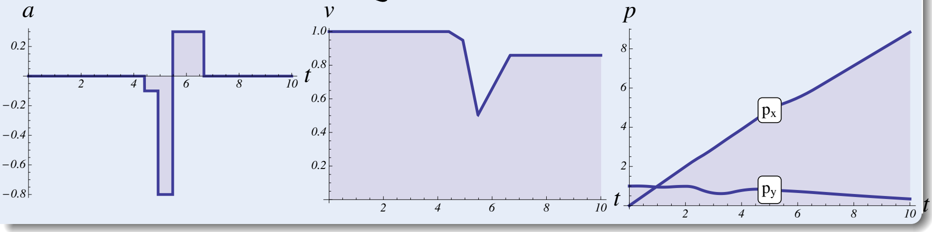
Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)







CPS Analysis: Robot Control

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Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
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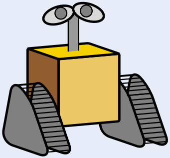
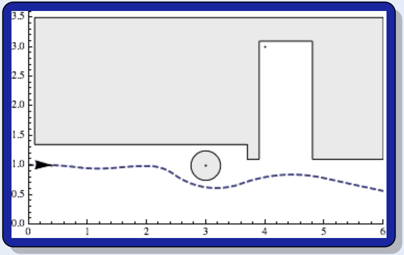
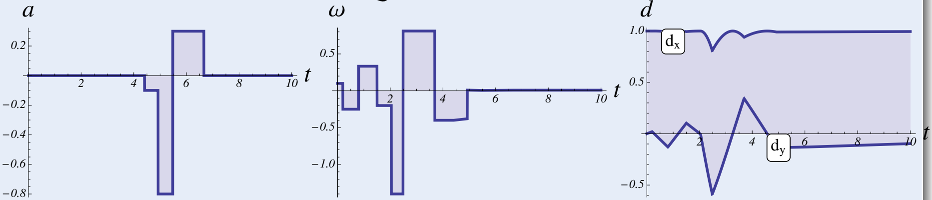




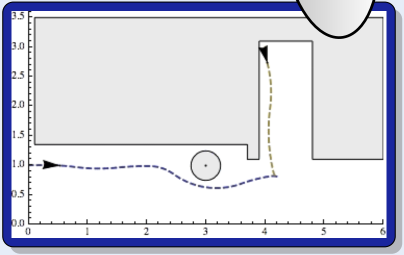


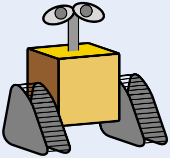
CPS Analysis: Robot Control

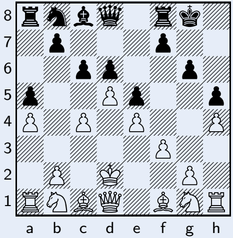
Challenge (Games)

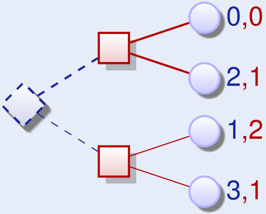
Game rules describing play evolution with both

- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)









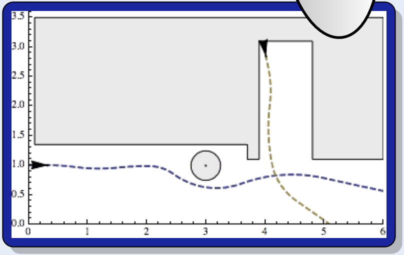
◇\□	Tr	Pl
Trash	1,2	0,0
Plant	0,0	2,1

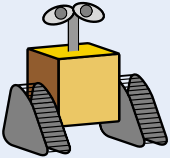
CPS Analysis: Robot Control

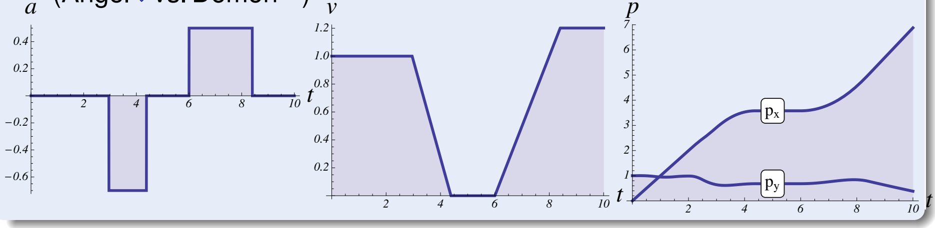
Challenge (Hybrid Games)

Game rules describing play evolution with

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)







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Differential Dynamic Logic dL: Syntax

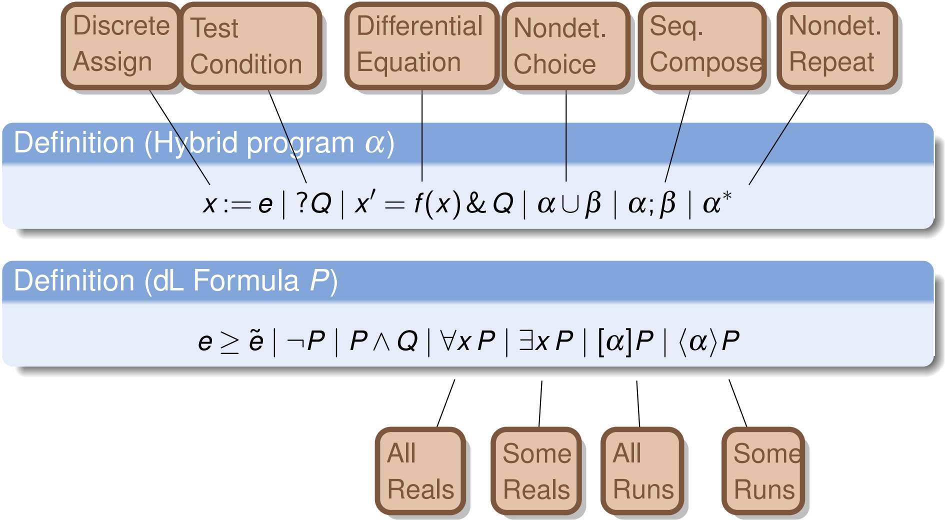
Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Differential Dynamic Logic dL: Syntax



Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

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All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic dL: Nondeterminism

Nondet. Choice

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Nondeterminism during HP runs

Differential Dynamic Logic dL: Nondeterminism

Differential Equation

Nondet. Choice

Nondet. Repeat

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All Choices

Some Choice

Differential Dynamic Logic dL: Nondeterminism

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

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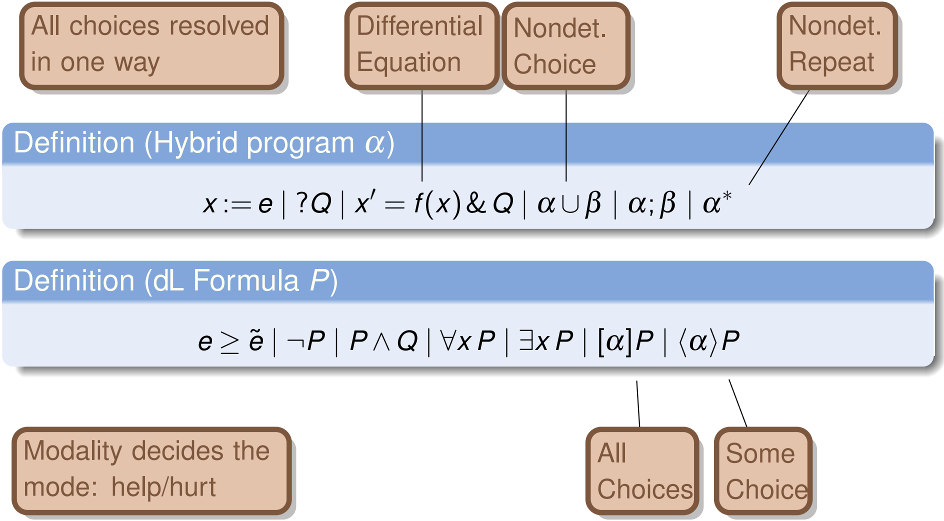
Modality decides the mode: help/hurt

All Choices

Some Choice

Differential Dynamic Logic dL: Nondeterminism



All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

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Modality decides the mode: help/hurt

All Choices

Some Choice

$[\alpha_1]\langle\alpha_2\rangle[\alpha_3]\langle\alpha_4\rangle P$ only fixed interaction depth

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

Let Angel be one player

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

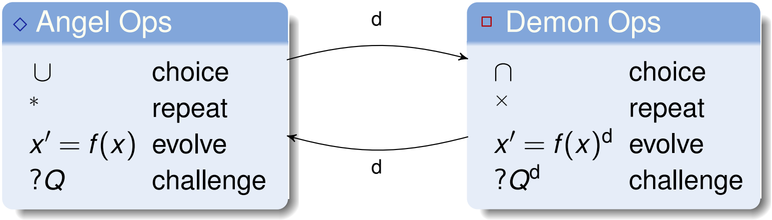
□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

Let Angel be one player

Let Demon be another player

Control & Dual Control Operators



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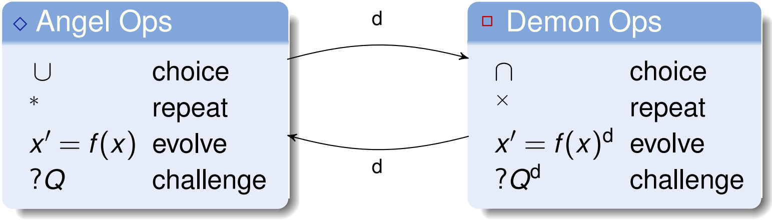
Duality operator $^{\mathrm{d}}$ passes control between players

Game Operators



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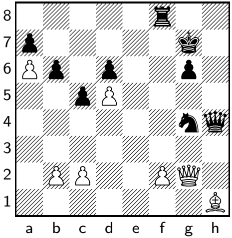
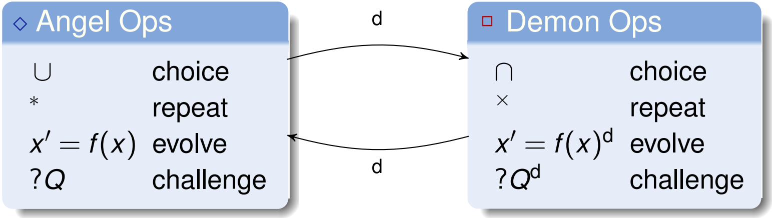
◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

Duality operator ^{d} passes control between players



Game Operators



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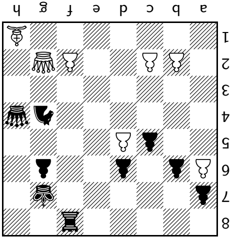
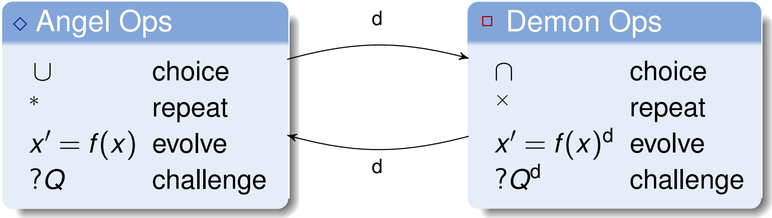


Game Operators



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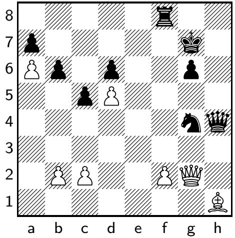
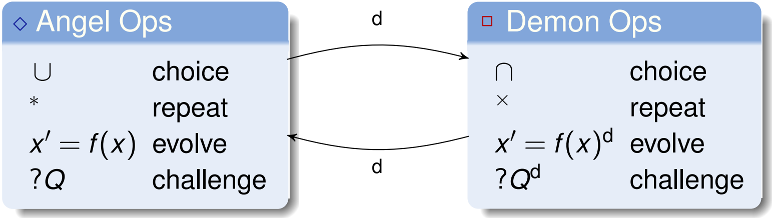


Definable Game Operators



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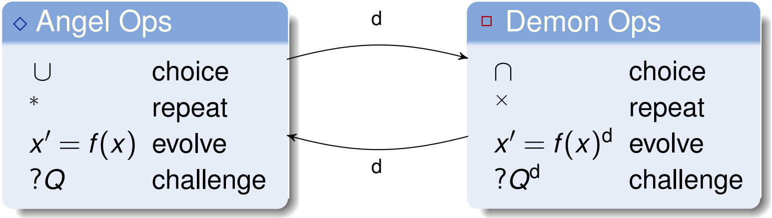
$^\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv \\
\text{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^\times &\equiv \\
(x' = f(x) \,\&\, Q)^d &\quad x' = f(x) \,\&\, Q \\
(x := e)^d &\quad x := e \\
?Q^d &\quad ?Q
\end{aligned}$$

Definable Game Operators



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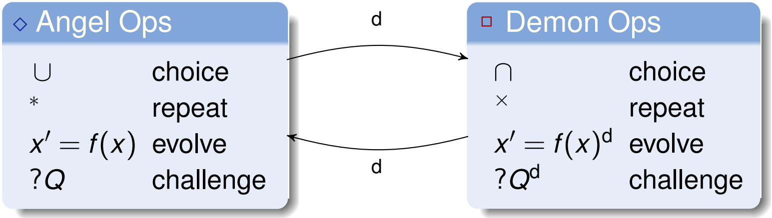
\text{while}(Q)\,\alpha &\equiv \\
\alpha\cap\beta &\equiv \\
\alpha^{\times} &\equiv \\
(x'=f(x)\,\&\,Q)^{d} &\quad x'=f(x)\,\&\,Q\\
(x:=e)^{d} &\quad x:=e\\
?Q^{d} &\quad ?Q
\end{aligned}$$

Definable Game Operators



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$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv$$

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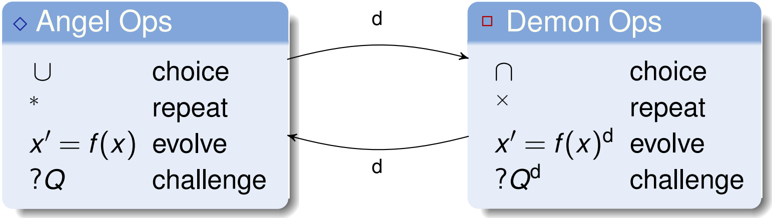
$$\begin{array}{rl} (x' = f(x)\,\&\,Q)^{\mathrm{d}} & x' = f(x)\,\&\,Q \\ (x := e)^{\mathrm{d}} & x := e \\ ?Q^{\mathrm{d}} & ?Q \end{array}$$

Definable Game Operators



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\alpha \cap \beta &\equiv\\
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(x' = f(x)\,\&\,Q)^{\mathrm{d}} &\quad x' = f(x)\,\&\,Q\\
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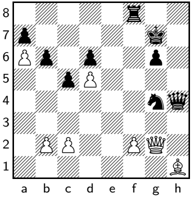
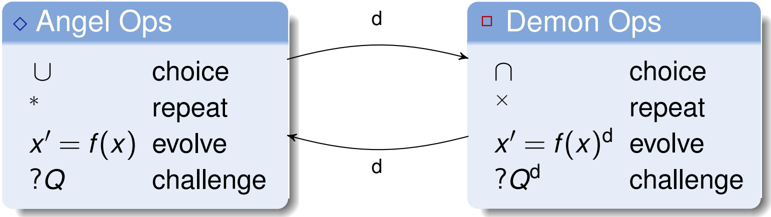


Definable Game Operators



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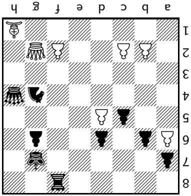
$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

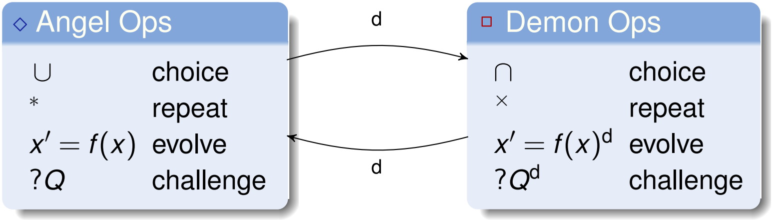
$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\alpha^\times \equiv$$

$$\begin{array}{rl} (x' = f(x)\,\&\,Q)^d & x' = f(x)\,\&\,Q \\ (x := e)^d & x := e \\ ?Q^d & ?Q \end{array}$$



Definable Game Operators



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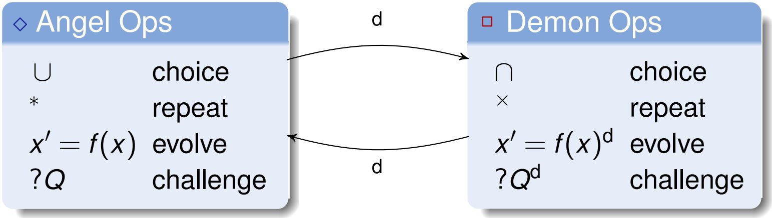
\mathrm{while}(Q)\,\alpha &\equiv (?Q;\alpha)^{*};?\neg Q\\
\alpha\cap\beta &\equiv (\alpha^{\mathrm{d}}\cup\beta^{\mathrm{d}})^{\mathrm{d}}\\
\alpha^{\times} &\equiv ((\alpha^{\mathrm{d}})^{*})^{\mathrm{d}}\\
(x' = f(x)\,\&\,Q)^{\mathrm{d}} &\quad x' = f(x)\,\&\,Q\\
(x := e)^{\mathrm{d}} &\quad x := e\\
?Q^{\mathrm{d}} &\quad ?Q
\end{aligned}$$

Definable Game Operators



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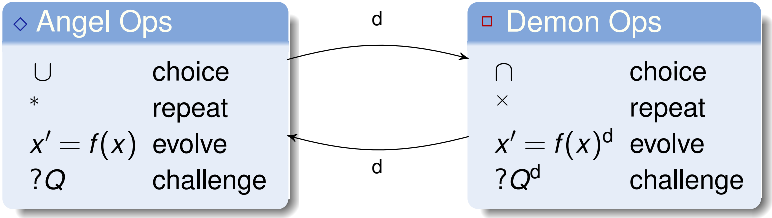
\text{while}(Q)\,\alpha &\equiv (?Q;\alpha)^*;?\neg Q\\
\alpha \cap \beta &\equiv (\alpha^d \cup \beta^d)^d\\
\alpha^\times &\equiv ((\alpha^d)^*)^d\\
(x' = f(x)\,\&\,Q)^d &\not\equiv x' = f(x)\,\&\,Q\\
(x := e)^d &\quad x := e\\
?Q^d &\quad ?Q
\end{aligned}$$

Definable Game Operators



◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

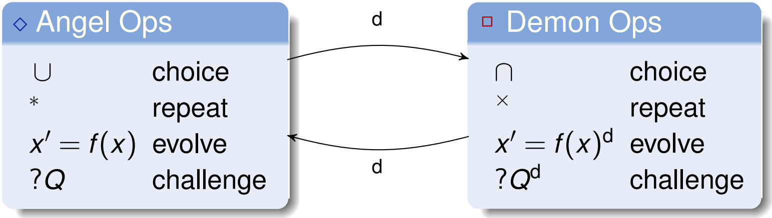
$$\alpha^\times \equiv ((\alpha^d)^*)^d$$

$$(x' = f(x)\,\&\,Q)^d \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^d \equiv x := e$$

$$?Q^d \qquad ?Q$$

Definable Game Operators



◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

$$
\begin{aligned}
\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta &\equiv (?Q;\alpha)\cup(?\neg Q;\beta)\\
\mathrm{while}(Q)\,\alpha &\equiv (?Q;\alpha)^{*};?\neg Q\\
\alpha\cap\beta &\equiv (\alpha^{d}\cup\beta^{d})^{d}\\
\alpha^{\times} &\equiv ((\alpha^{d})^{*})^{d}\\
(x' = f(x)\,\&\,Q)^{d} &\not\equiv x' = f(x)\,\&\,Q\\
(x := e)^{d} &\equiv x := e\\
?Q^{d} &\not\equiv ?Q
\end{aligned}
$$

Outline

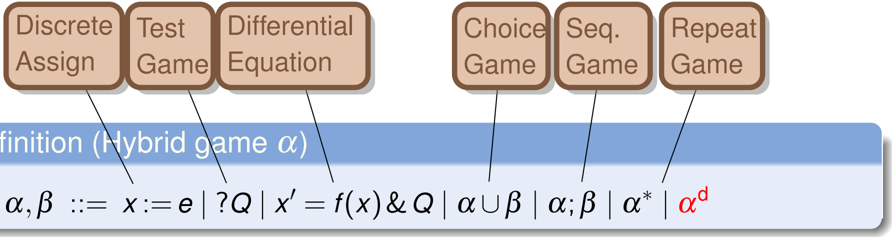
1. Learning Objectives
2. Motivation
3. A Gradual Introduction to Hybrid Games
   - Choices & Nondeterminism
   - Control & Dual Control
   - Demon's Derived Controls
4. Differential Game Logic
   - Syntax of Hybrid Games
   - Syntax of Differential Game Logic Formulas
   - Examples
   - Push-around Cart
   - Robot Dance
   - Example: Robot Soccer
5. An Informal Operational Game Tree Semantics
6. Summary

Hybrid Games: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Hybrid Games: Syntax



Definition (Hybrid game α)

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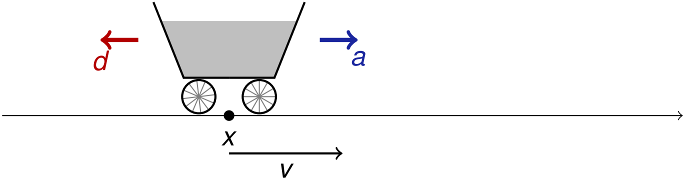
Hybrid Games: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

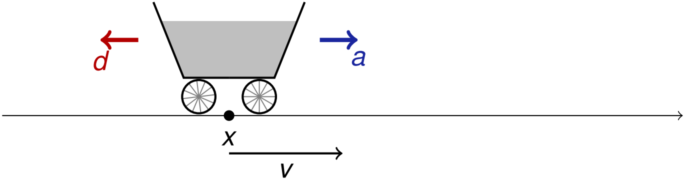
Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Example: Push-around Cart

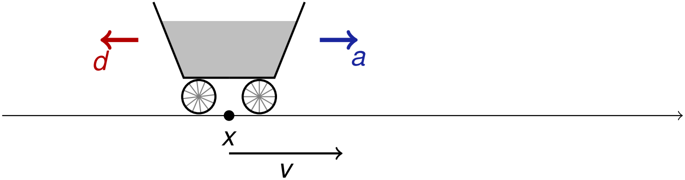


Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cup d:=-1)^{\mathrm{d}};\{x'=v,v'=a+d\}\big)^{*}$$

Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cup d:=-1)^{d};\{x'=v,v'=a+d\}\big)^{*}$$

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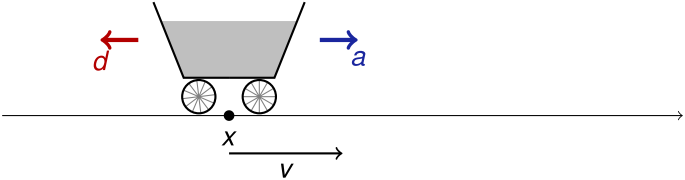
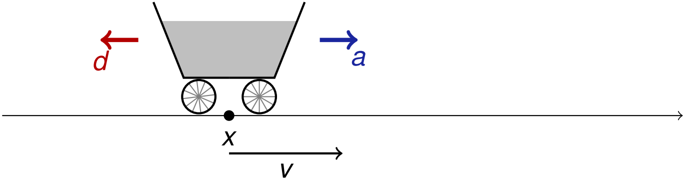
Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\big)^*$$

$$\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$

Example: Push-around Cart

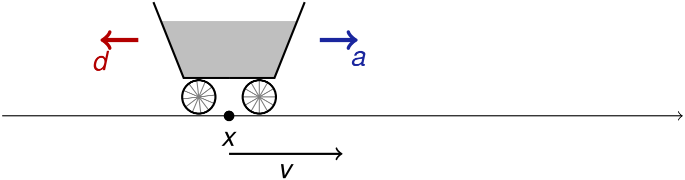


$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v, v'=a+d\}\big)^*$$

$$\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*$$

HP $\big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*$

Example: Push-around Cart



$$\left((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\right)^*$$

$$\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\right)^*$$

$$\text{HP}\ \left((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\right)^*$$

Hybrid systems can't say that a is Angel's choice and d is Demon's

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha,\beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P,Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha] P$$

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

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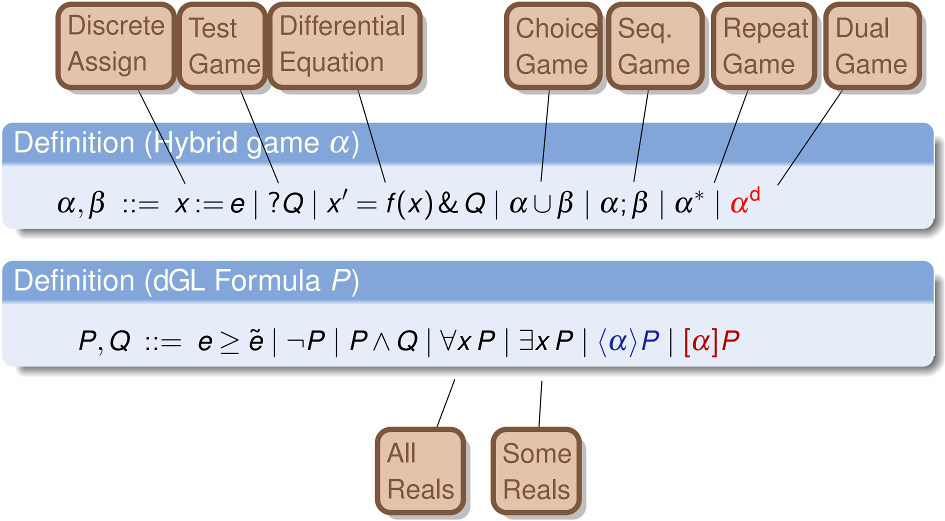
All Reals | Some Reals

Differential Game Logic: Syntax



Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Dual Game

Definition (Hybrid game α)

$$\alpha,\beta ::= x:=e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P,Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha]P$$

All Reals

Some Reals

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

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$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x \, P \mid \exists x \, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals | Angel Wins

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \color{red}{\alpha^d}$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \color{blue}{\langle \alpha \rangle P} \mid \color{red}{[\alpha] P}$$

All Reals | Some Reals | Angel Wins | Demon Wins

Simple Examples

$$\langle (x := x+1; (x'=1)^{d} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

$$\langle (x := x+1; (x'=1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \leq x < 1)$$

Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^{\mathrm{d}} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

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Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

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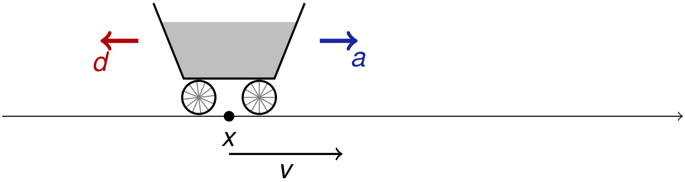
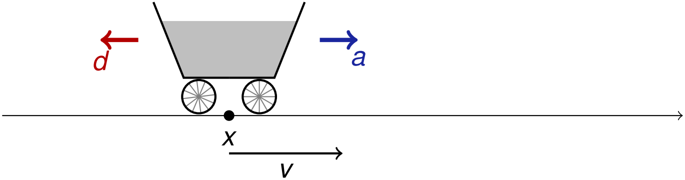
Example: Push-around Cart



$$v \geq 1 \rightarrow$$

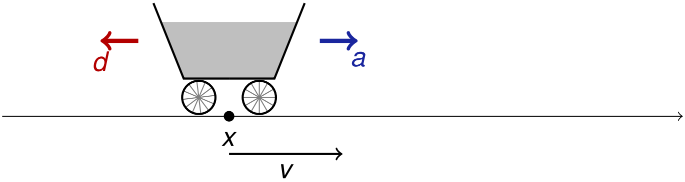
$$\left[\left((d := 1 \cup d := -1)^{\mathrm{d}}; (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^{*}\right] v \geq 0$$

Example: Push-around Cart



$$\vDash v \geq 1 \rightarrow$$
$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

Example: Push-around Cart



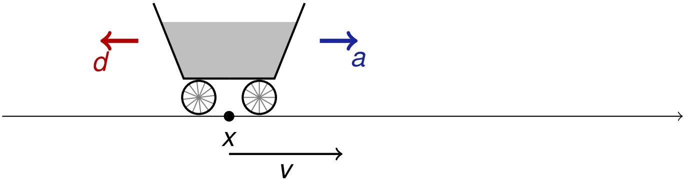
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\right)^*\right] v \geq 0$$

$x \geq 0 \wedge v \geq 0 \rightarrow$

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\right)^*\right] x \geq 0$$

Example: Push-around Cart



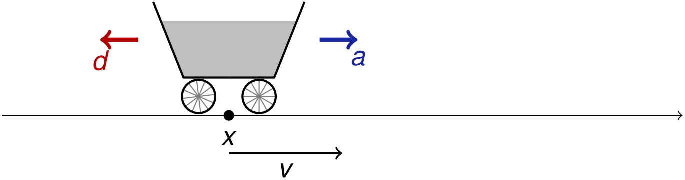
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\vDash x \geq 0 \wedge v \geq 0 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, x \geq 0$$

Example: Push-around Cart



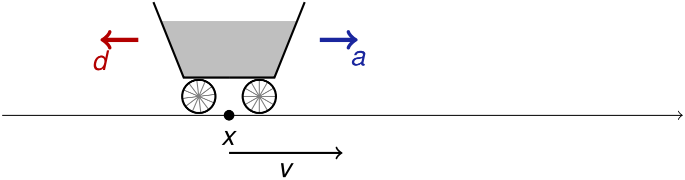
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$$x \geq 0 \rightarrow$$

$$\langle ((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^* \rangle x \geq 0$$

Example: Push-around Cart



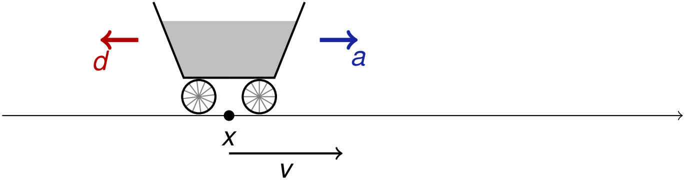
$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

$$\Big[\big((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\}\big)^*\Big] v \geq 0$$

$\vDash x \geq 0 \qquad \rightarrow$ boring by skip

$$\Big\langle\big((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\}\big)^*\Big\rangle x \geq 0$$

Example: Push-around Cart

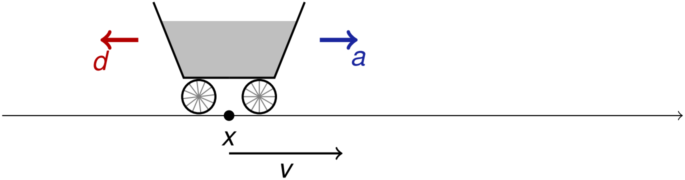


$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\Big[\big((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\}\big)^*\Big]v \geq 0$$

$$\Big\langle\big((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\}\big)^*\Big\rangle x \geq 0$$

Example: Push-around Cart



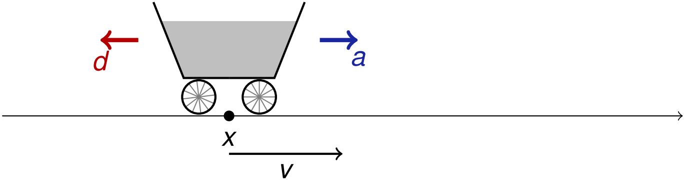
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\not\vDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

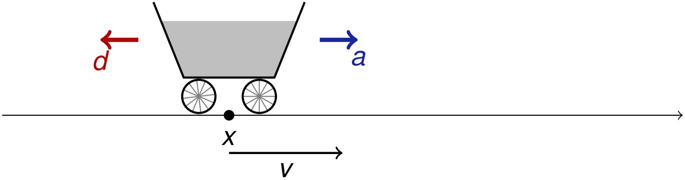
$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\nvDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

$$\langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

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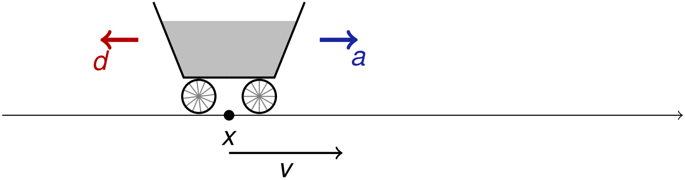
Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

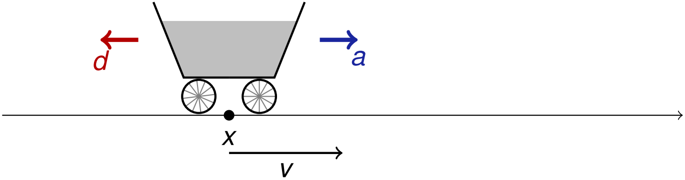
$\nvDash$ counterstrategy $d := -1$

$$\left\langle\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

$$\vDash \left\langle\left((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

$$\left\langle\left((d := 2 \cap d := -2); (a := 2 \cup a := -2); t := 0; \{x' = v, v' = a + d, t' = 1 \,\&\, t \leq 1\}\right)^*\right\rangle x^2 \geq 100$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\nvDash$ counterstrategy $d := -1$

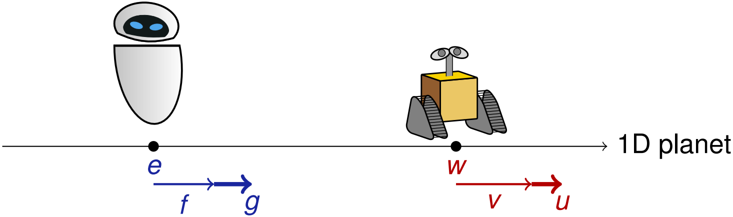
$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

$\vDash \langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle x \geq 0$

$\vDash \langle((d := 2 \cap d := -2); (a := 2 \cup a := -2);$ $a := d$ then $a := 2 \operatorname{sign} v$

$$t := 0; \{x' = v, v' = a + d, t' = 1 \,\&\, t \leq 1\})^*\rangle x^2 \geq 100$$

Example: WALL·E and EVE Robot Dance

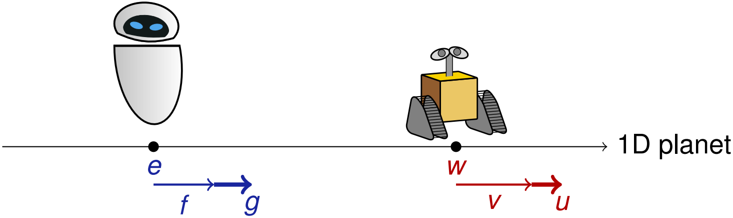


$$
\begin{aligned}
&(w-e)^2 \leq 1 \land v = f \rightarrow \\
&\langle ((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1\}^{d} \\
&)^{\times} \rangle \, (w-e)^2 \leq 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

Example: WALL·E and EVE Robot Dance and the World

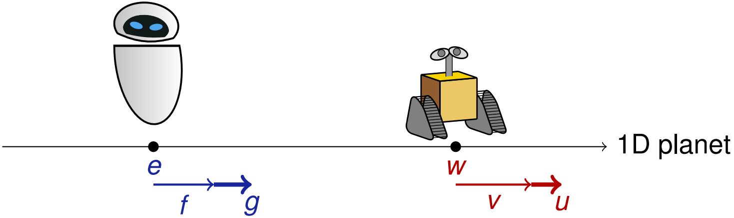


$$(w-e)^2 \le 1 \land v = f \rightarrow$$
$$\langle \big((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d}$$
$$\big)^{\times} \rangle \, (w-e)^2 \le 1$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u and world time

Example: WALL·E and EVE

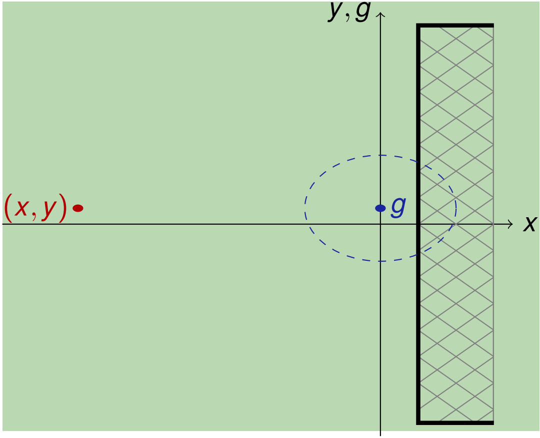


$$\begin{aligned}
&(w-e)^2 \le 1 \land v = f \to \\
&\Big[\Big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\} \\
&\Big)^{\times}\Big]\, (w-e)^2 > 1
\end{aligned}$$

WALL·E at w plays Demon's part controlling u and world time

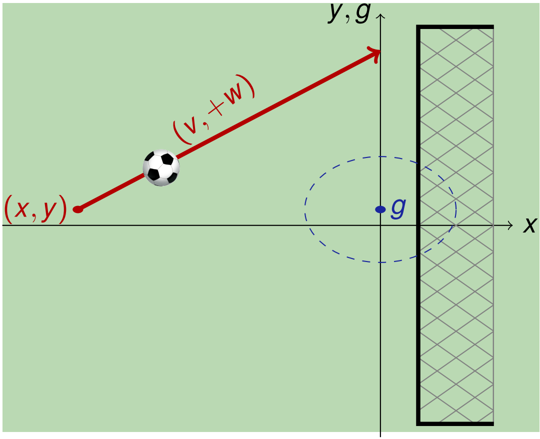
EVE at e plays Angel's part controlling g

Example: Goalie in Robot Soccer



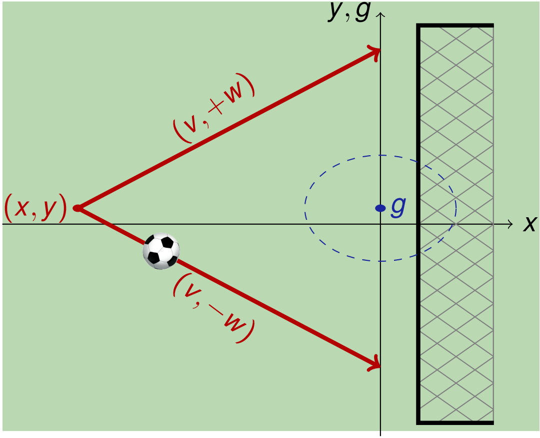
$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\Big\langle (w := +w \cap w := -w);$$
$$\Big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\Big)^* \Big\rangle \, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



$$x < 0 \land v > 0 \land y = g \rightarrow \\ \langle (w := +w \cap w := -w); \\ \big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \rangle\, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



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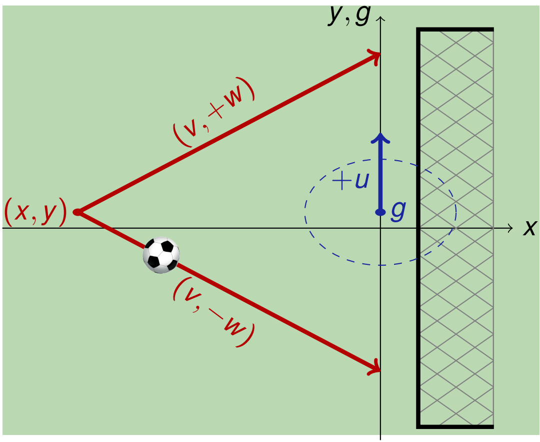

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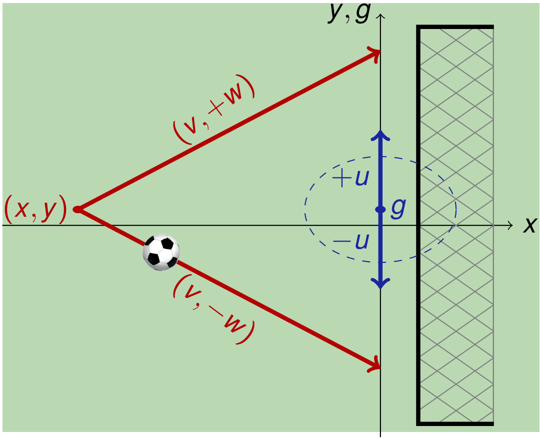
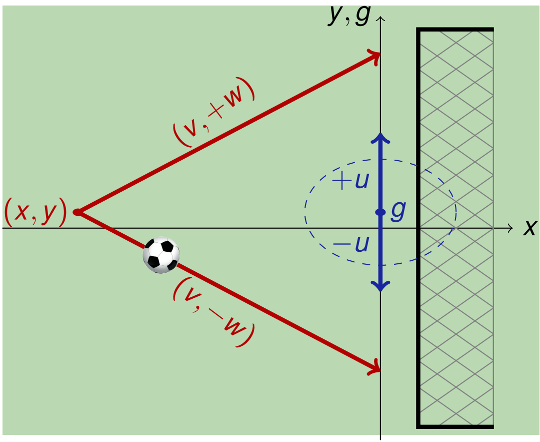


$$x < 0 \land v > 0 \land y = g \rightarrow \\ \big\langle (w := +w \cap w := -w); \\ \big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \big\rangle\, x^2 + (y - g)^2 \le 1$$

Example: Goalie in Robot Soccer



Goalie's Secret

$$\left(\frac{x}{v}\right)^2 (u-w)^2 \leq 1 \land$$

$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\big\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^*\big\rangle\, x^2 + (y-g)^2 \leq 1$$

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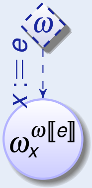
5. **An Informal Operational Game Tree Semantics**
6. Summary

Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)

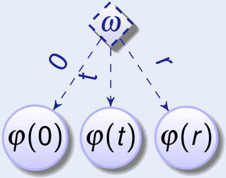
$x := e$



Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)

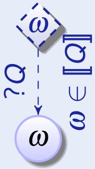
$x' = f(x) \,\&\, Q$



Differential Game Logic: Operational Semantics

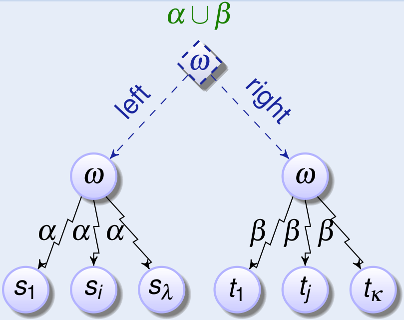
Definition (Hybrid game α: operational semantics)

$?Q$



Differential Game Logic: Operational Semantics

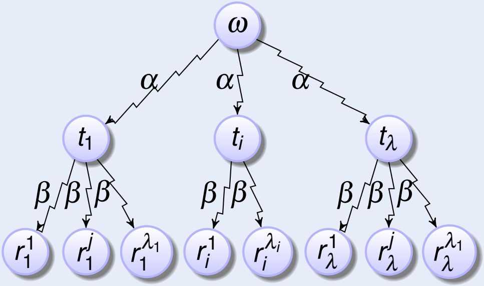
Definition (Hybrid game α: operational semantics)



Differential Game Logic: Operational Semantics

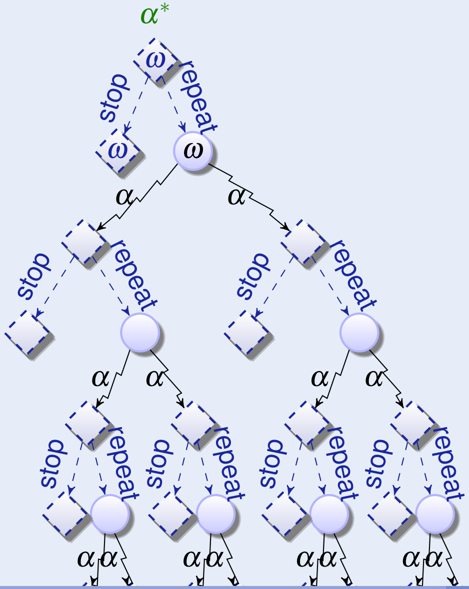
Definition (Hybrid game α: operational semantics)

$\alpha;\beta$



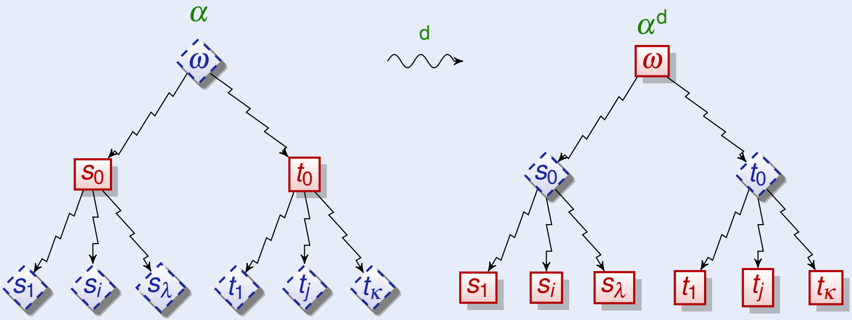
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



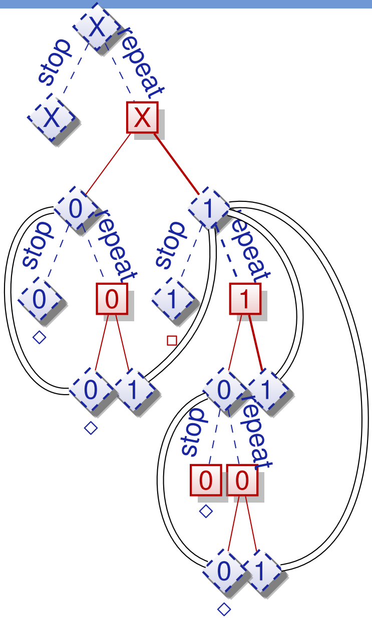
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



Filibusters

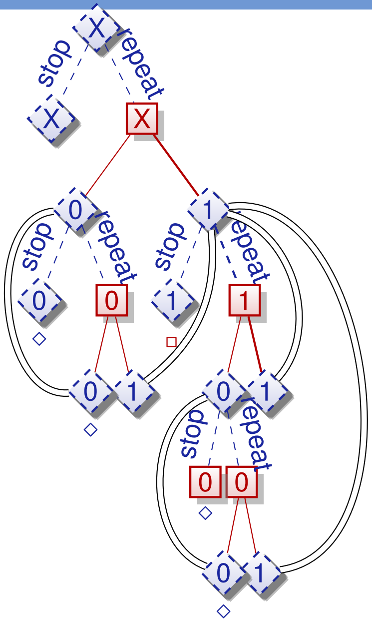
$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$



Filibusters & The Significance of Finitude

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



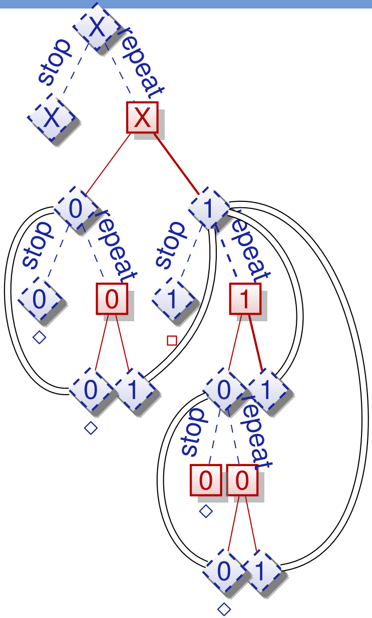
Filibusters & The Significance of Finitude

$$\langle (x' = 1^{d}; x := 0)^{*} \rangle x = 0$$

$$\langle (x := 0; x' = 1^{d})^{*} \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^{*} \rangle x = 0$$

$$\overset{\text{wfd}}{\rightsquigarrow} \text{ false unless } x = 0$$



Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

$\langle (x' = 1^{d}; x := 0)^{*} \rangle x = 0$

$\langle (x := 0; x' = 1^{d})^{*} \rangle x = 0$

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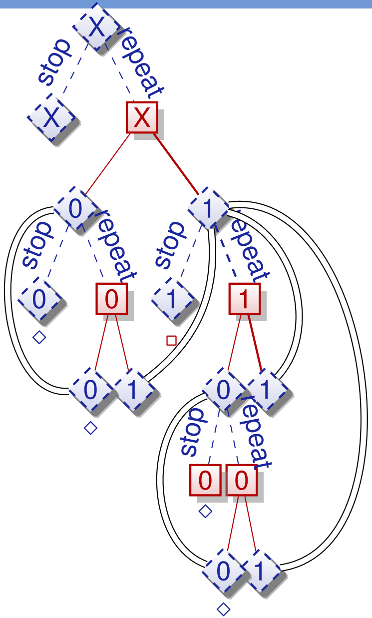
$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

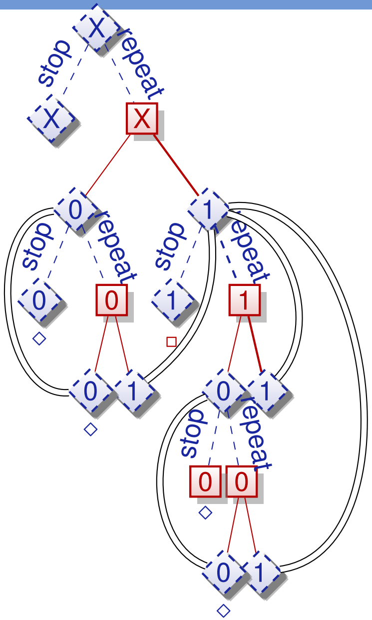
$$\langle (x' = 1^{\mathrm{d}}; x := 0)^* \rangle x = 0$$

$$\langle (x := 0; x' = 1^{\mathrm{d}})^* \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever



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Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Dual Game

Definition (Hybrid game α)

$$\alpha, \beta \;::=\; x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \;::=\; e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

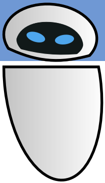
All Reals

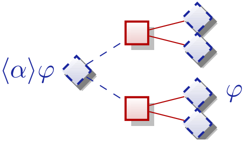
Some Reals

Angel Wins

Demon Wins

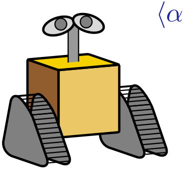
Summary





differential game logic

$$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{\mathsf{d}}$$

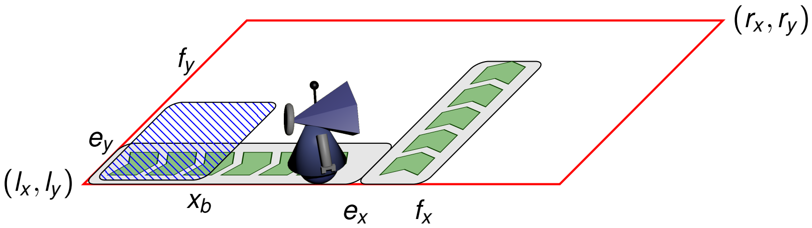


- Differential game logic
- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Operational semantics (informally)

Outline

7 Example: Robot Factory

Example: Robot Factory Decentralized Automation



Model

- (x, y) robot coordinates
- (v_x, v_y) velocities
- conveyor belts may instantaneously increase robot's velocity by (c_x, c_y)

Primary objectives of the robot

- Leave ▨ within time ε
- Never leave outer □

Challenges

- Distributed, physical environment
- Possibly conflicting secondary objectives

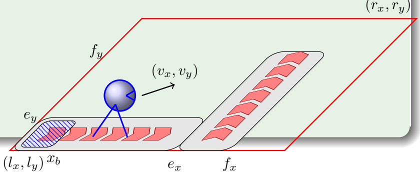
Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \land y < e_y \land \mathit{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathit{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \land y \le f_y \land \mathit{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathit{eff}_2 := 0)\big);$$

$$\Big)^*$$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$\Big(\big(?true \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0)$ // belt

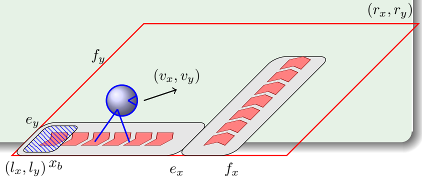
$\cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$

$(a_x := *;\ ?(-A \le a_x \le A);$

$a_y := *;\ ?(-A \le a_y \le A);$ // “independent” robot acceleration

$t_s := 0)^d;$

$\Big)^*$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

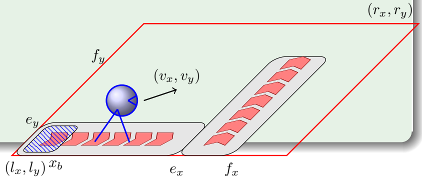
$$(a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0)^d;$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon);$$

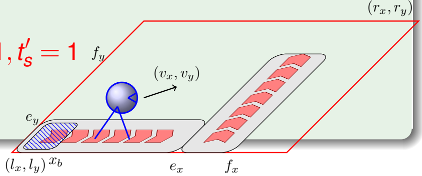
$$\Big)^*$$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$
\begin{aligned}
&\Big(\big(?\mathit{true} \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) && \text{// belt}\\
&\qquad\qquad \cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);\\
&\ (a_x := *;\ ?(-A \le a_x \le A);\\
&\ \ a_y := *;\ ?(-A \le a_y \le A); && \text{// “independent” robot acceleration}\\
&\ \ t_s := 0)^{\mathrm{d}};\\
&\big((x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1 \ \&\ t_s \le \varepsilon);\\
&\ \cap (?(a_x v_x \le 0 \land a_y v_y \le 0)^{\mathrm{d}}; && \text{// brake}\\
&\ \ \text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi}; && \text{// per direction: no time lock}\\
&\ \ \text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi};\\
&\ \ (x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\\
&\ \ \&\ t_s \le \varepsilon \land a_x v_x \le 0 \land a_y v_y \le 0))\big)\Big)^{*}
\end{aligned}
$$



Robot Factory Automation (*RF*)

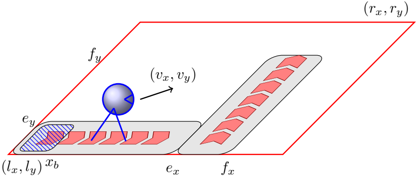
Proposition (Robot stays in □)

$$\models (x = y = 0 \land v_x = v_y = 0 \land \text{▸ Controllability Assumptions}\;)$$
$$\to [RF](x \in [l_x, r_x] \land y \in [l_y, r_y])$$

Proposition (Stays in □ and leaves ▨ on time)

$RF|_x$: RF projected to the x-axis

$$\models (x = 0 \land v_x = 0 \land \text{▸ Controllability Assumptions}\;)$$
$$\to [RF|_x](x \in [l_x, r_x] \land (t \geq \varepsilon \to x \geq x_b))$$



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