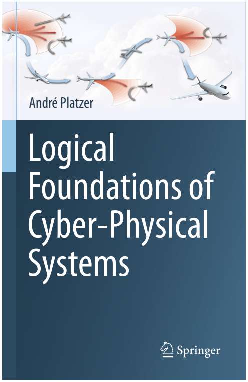
15: Winning Strategies & Regions

Logical Foundations of Cyber-Physical Systems



Stefan Mitsch

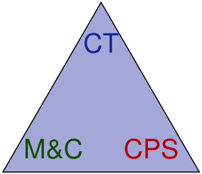
Carnegie Mellon University
Computer Science Department

Outline

Outline

1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

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"Angel has Wings $\langle\alpha\rangle$"

All Reals | Some Reals | Angel Wins | Demon Wins

Outline

Differential Game Logic: Denotational Semantics

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Differential Game Logic: Denotational Semantics

Only for HPs. No interactive play!

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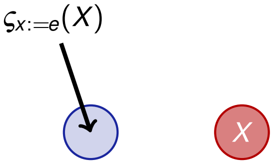

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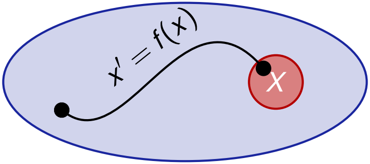

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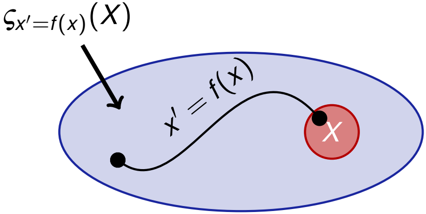

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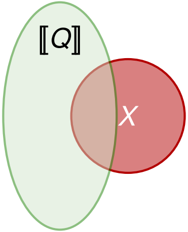

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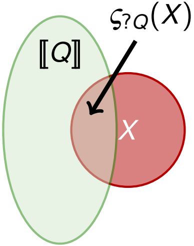
$$\varsigma_{?Q}(X) = [\![Q]\!] \cap X$$



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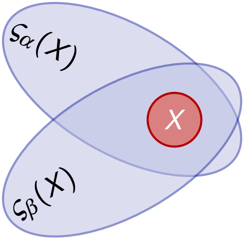

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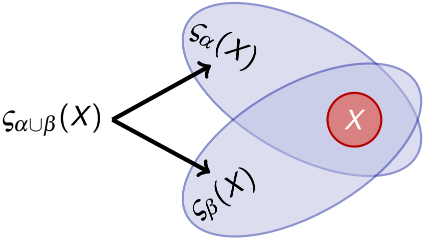

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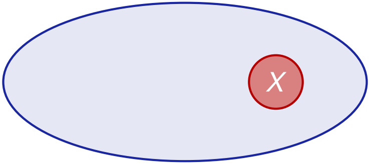

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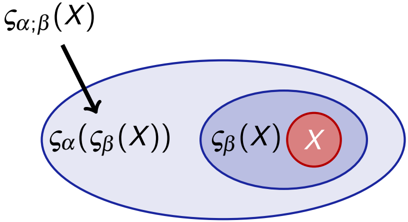

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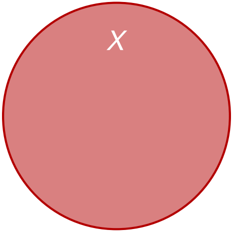

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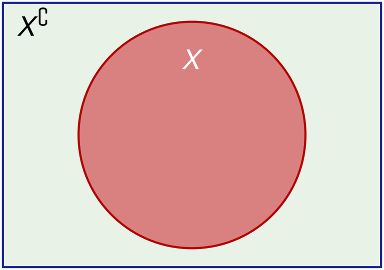

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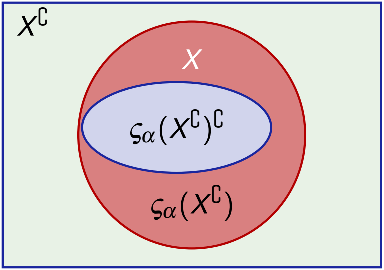

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Definition (Hybrid game α: denotational semantics)

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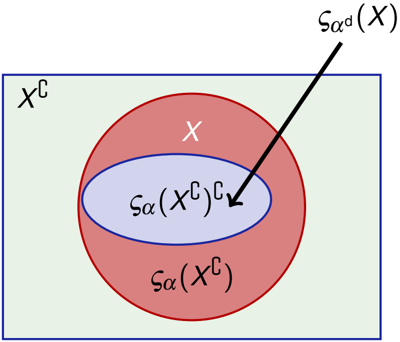

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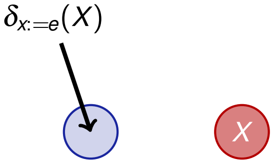

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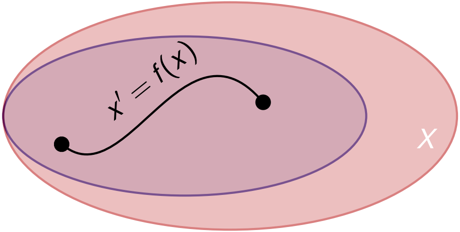

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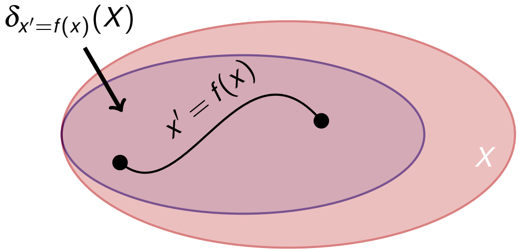

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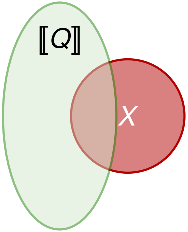

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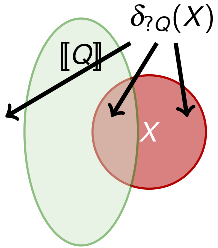
$$\delta_{?Q}(X) = [\![Q]\!]^\complement \cup X$$



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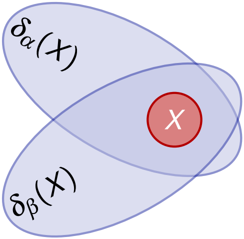

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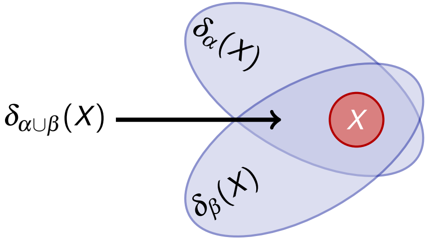

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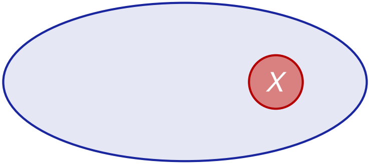

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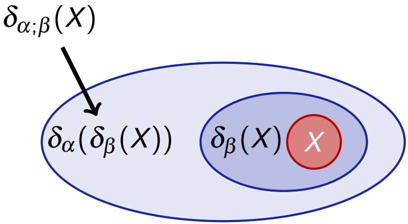

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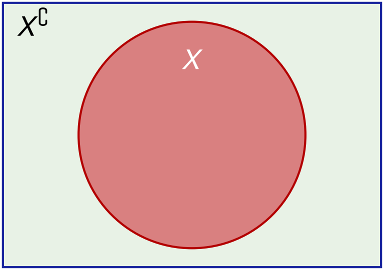

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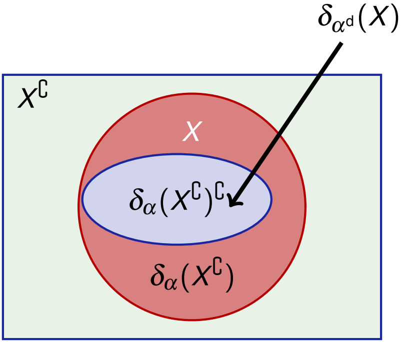

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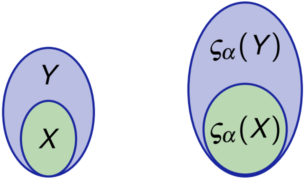
Monotonicity

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$



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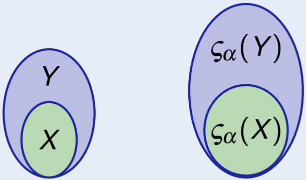
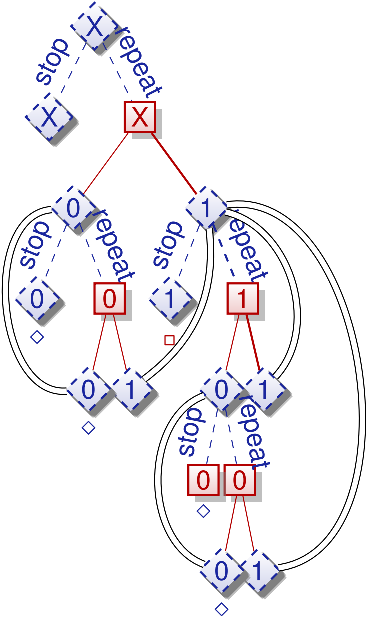


Outline

Filibusters & The Significance of Finitude

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) =$

Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$

$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]$ where $\alpha^{n+1} \equiv \alpha^n;\alpha$ $\quad\alpha^0 \equiv ?true$ for HP α

Semantics of Repetition

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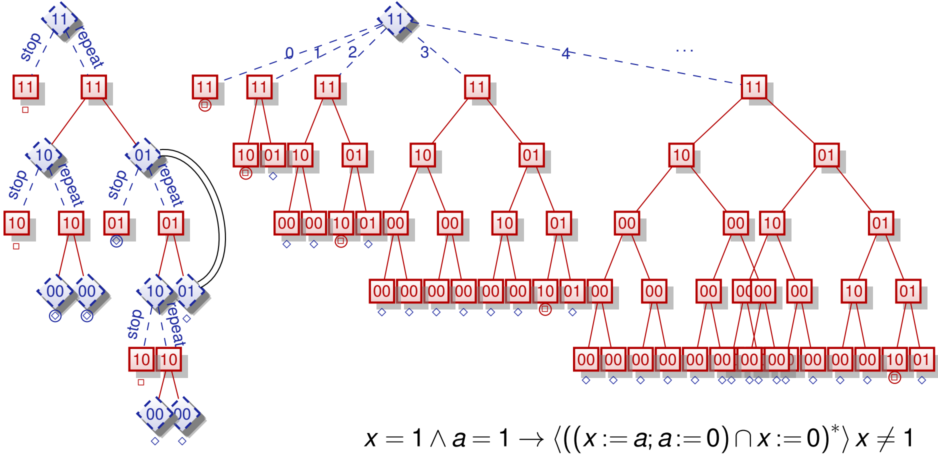
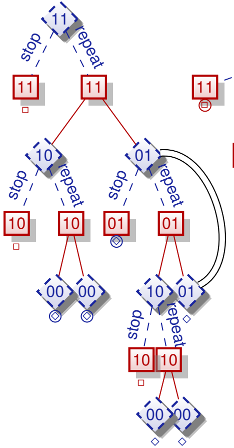
$$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$$

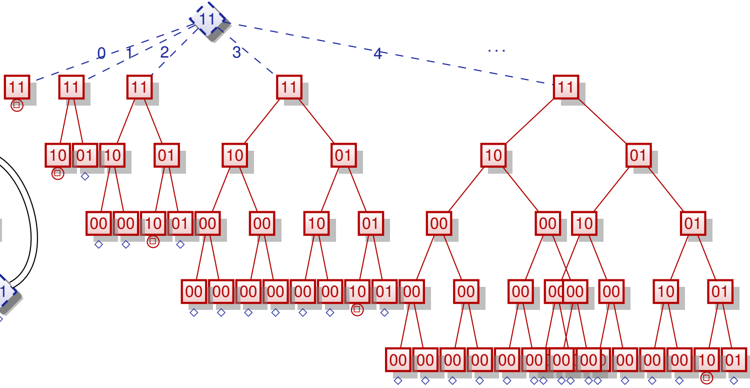


$$x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ advance notice semantics?

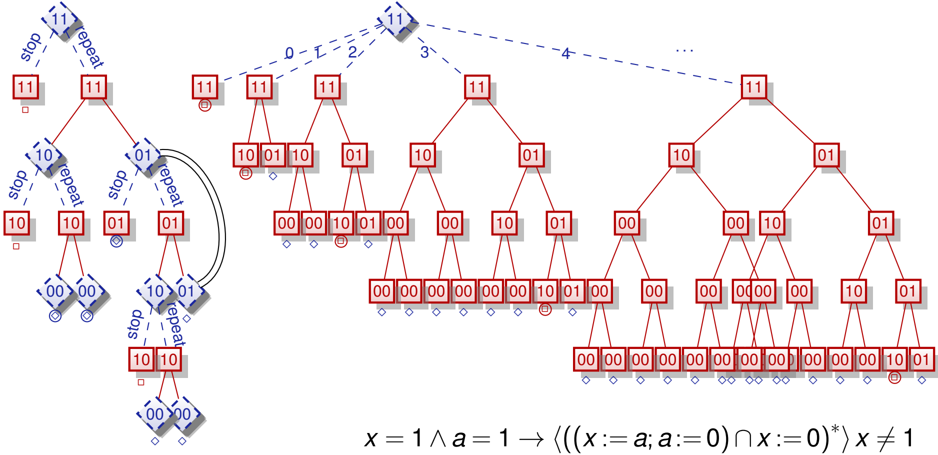




$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ too hard to predict all iterations!



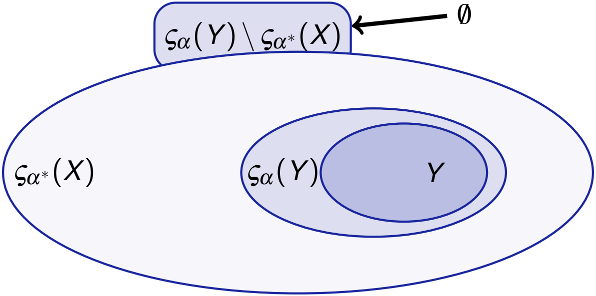
$x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle\, x \neq 1$

+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$

$$\varsigma_\alpha^0(X) \stackrel{\text{def}}{=} X$$

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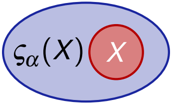

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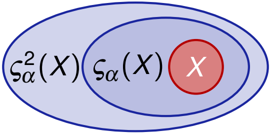

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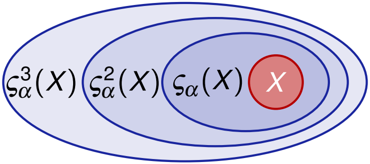

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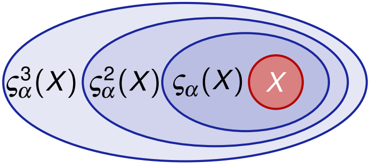

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Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \le x < 1)$

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$$\varsigma_\alpha^\omega([0,1)) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n([0,1)) = [0,\infty) \neq \mathbb{R}$$

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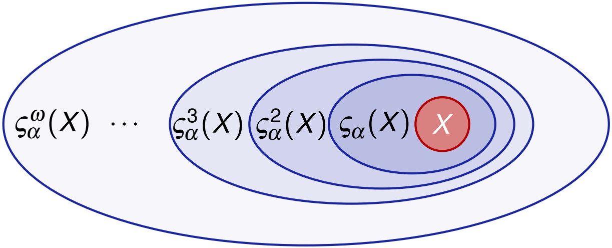

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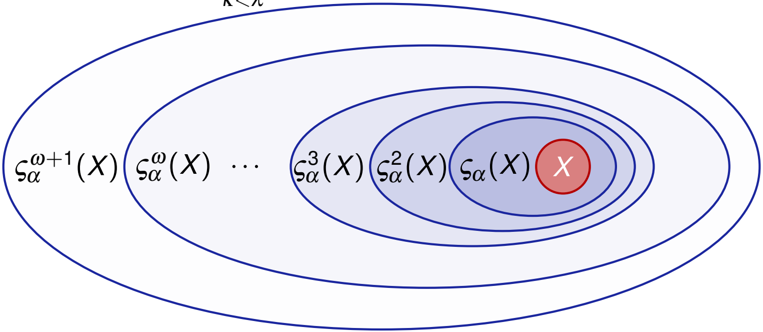
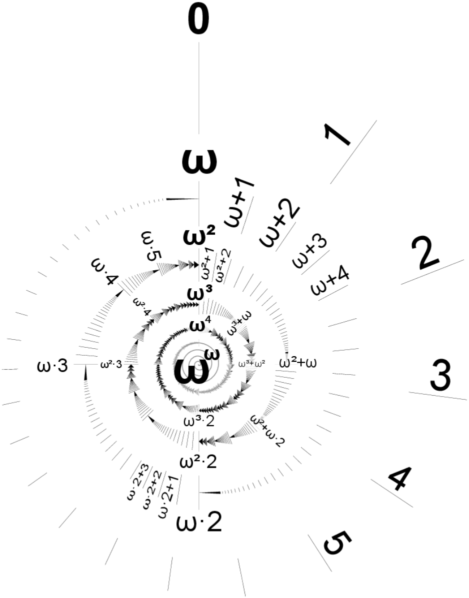


Theorem

Hybrid game closure ordinal $> \omega^\omega$



Expedition: Ordinal Arithmetic

$$\iota + 0 = \iota$$

$$\iota + (\kappa{+}1) = (\iota + \kappa) + 1 \qquad \text{successor } \kappa{+}1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \qquad \text{limit } \lambda$$

$$\iota \cdot 0 = 0$$

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$$\iota^0 = 1$$

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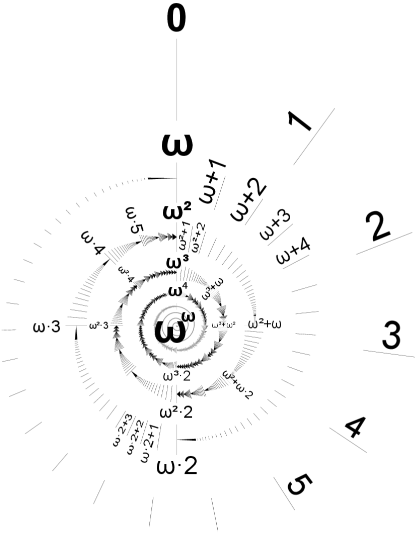
$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$



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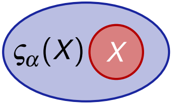

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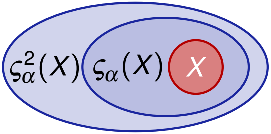

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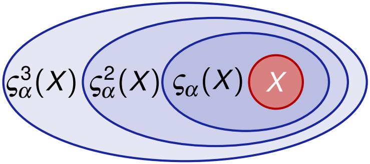

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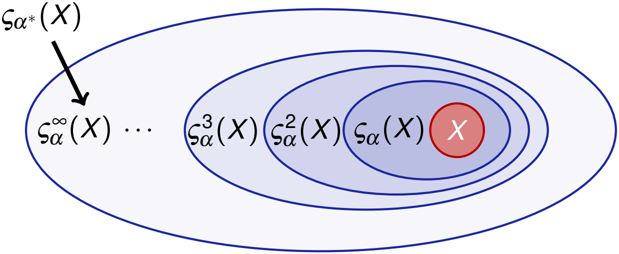
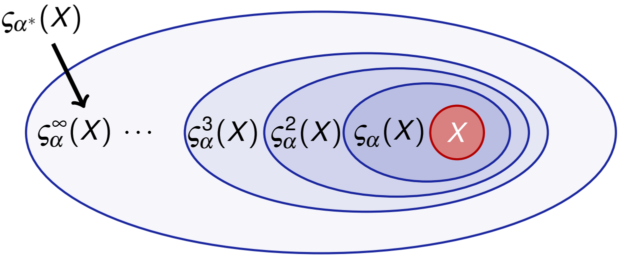


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ requires transfinite patience



The Power of Implicit Definitions

Implicit Definitions

The advantages of implicit definition over construction are roughly those of theft over honest toil.

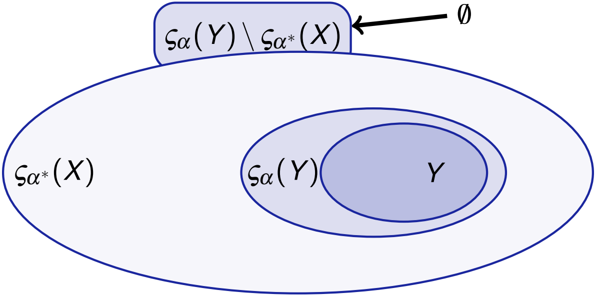
— Bertrand Russell

+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



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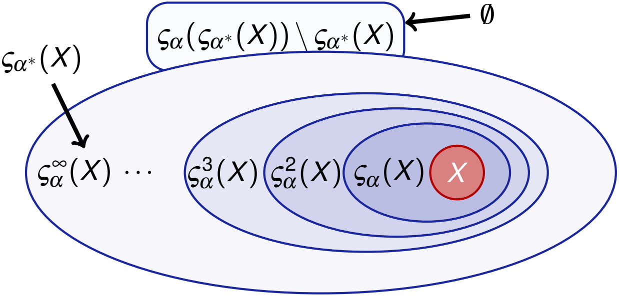
- Still too small: $X \subseteq Z$ since Angel may decide not to repeat

Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

$X \cup \varsigma_\alpha(Z) \subseteq Z$ for the winning region $Z \stackrel{\text{def}}{=} \varsigma_{\alpha^*}(X)$



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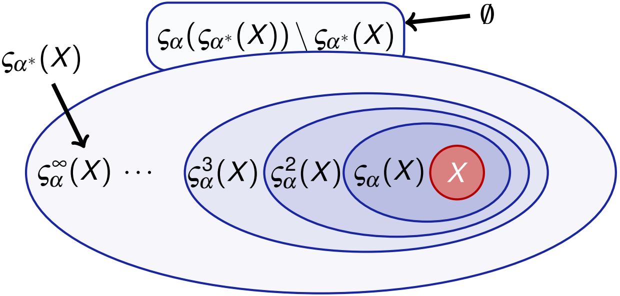

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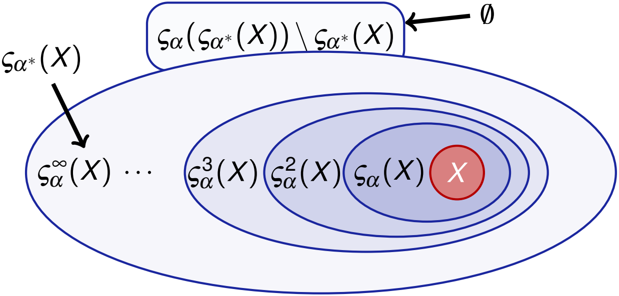

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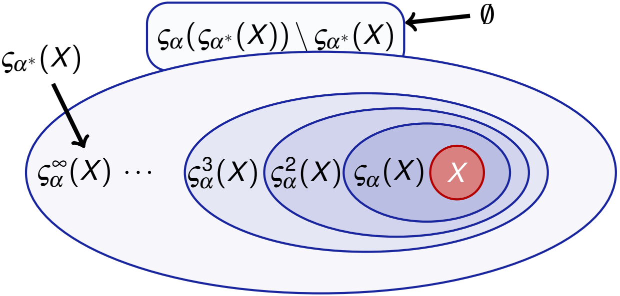

- Which Z is the right one?
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- Existence: $Z = \mathscr{S}$ but that's too big and independent of α

Comparing (Pre-)Fixpoints

Lemma ( )

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
$$X \cup \varsigma_\alpha(Z) \subseteq Z$$

are pre-fixpoints, then

Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
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Proof.

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□

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Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!

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So: repetition semantics is the smallest pre-fixpoint (well-founded)

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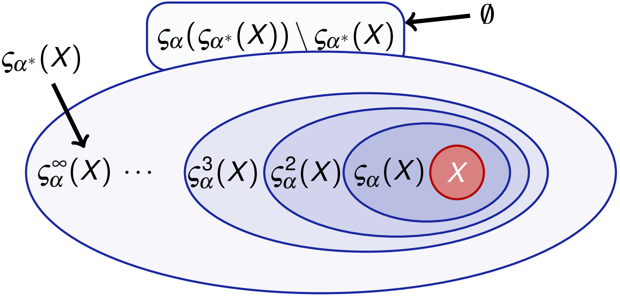

$X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$

$\varsigma_{\alpha^*}(X)$ intersection of solutions

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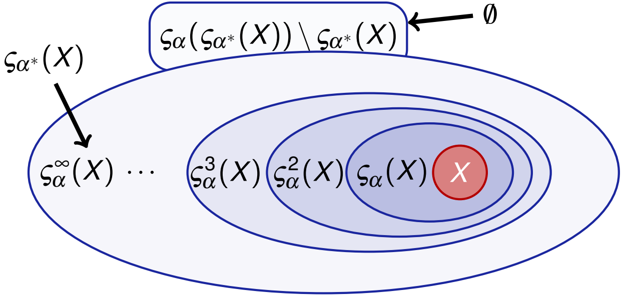

$Z \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$ — $\varsigma_{\alpha^*}(X)$ intersection of solutions

$\varsigma_\alpha(Z) \subseteq \varsigma_\alpha(\varsigma_{\alpha^*}(X))$ — by mon since $Z \subseteq \varsigma_{\alpha^*}(X)$

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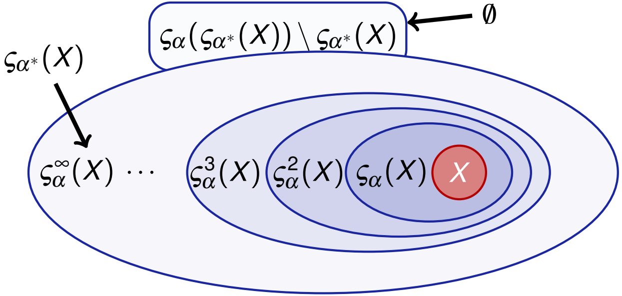

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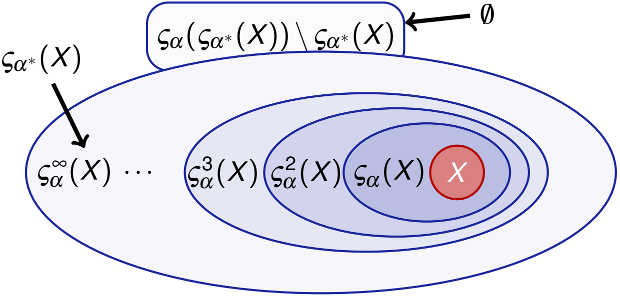

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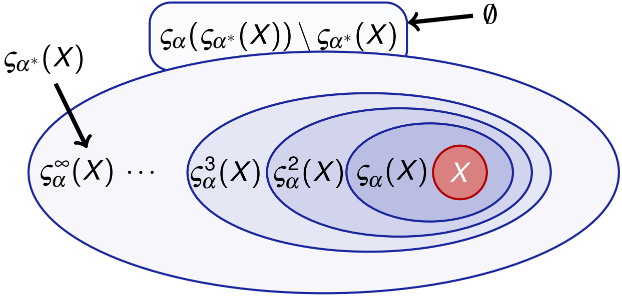

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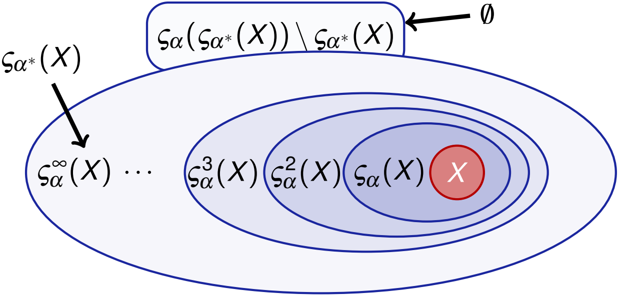

$Z \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$ — $\varsigma_{\alpha^*}(X)$ intersection of solutions

$X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) = Z$ — by mon since $Z \subseteq \varsigma_{\alpha^*}(X)$

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Semantics of Repetition

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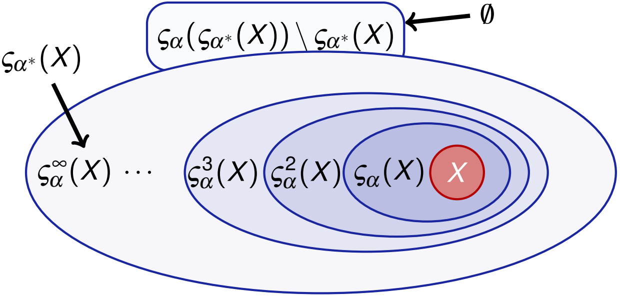

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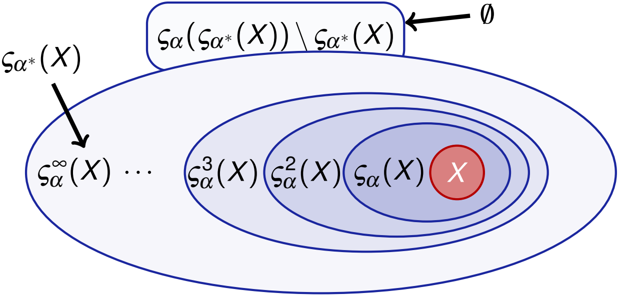
Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) = Z\} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski



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Outline

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathscr{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
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\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
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Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

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\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathscr{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
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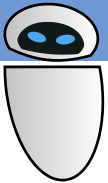
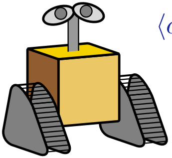
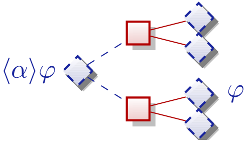
Summary



differential game logic

dGL $= \text{GL} + \text{HG} = \text{dL} + {}^{d}$





- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint

Next chapter

1. Axiomatics
2. How to win and prove hybrid games

André Platzer.
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URL: http://www.springer.com/978-3-319-63587-3,
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André Platzer.
Differential game logic.
ACM Trans. Comput. Log., 17(1):1:1–1:51, 2015.
doi:10.1145/2817824.