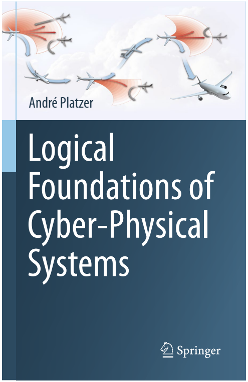
20: Virtual Substitution & Real Equations

Logical Foundations of Cyber-Physical Systems



Stefan Mitsch

Carnegie Mellon University
Computer Science Department

Outline

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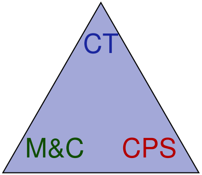
rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles



analytic complexity
modeling tradeoffs

verifying CPS at scale

Outline

1. Learning Objectives
2. Framing the Miracle
3. Quantifier Elimination
   - Homomorphic Normalization for QE
   - Term Substitutions for Linear Equations
4. Square Root $\sqrt{\cdot}$ Virtual Substitution for Quadratics
   - Square Root Algebra
   - Virtual Substitutions of Square Roots
   - Example
5. Summary

Evaluating Real Arithmetic Formulas

$x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2$

Evaluating Real Arithmetic Formulas

When $\omega(x) = 2$

$$\omega[\![x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2]\!]$$

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When $\nu(x) = -1$

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Evaluating Real Arithmetic Formulas

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Are the following formulas valid, i.e., true in all states?

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$$\forall x\,(x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2)$$
$$\exists x\,(x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2)$$

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Evaluating Real Arithmetic Formulas

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Framing the Miracle: Quiz

Is validity of formulas
decidable/semidecidable/undecidable/not semidecidable for:

1. Propositional logic [no variables]

✓ FOL$[p, f, \dots]$ uninterpreted

2. FOL$_{\mathbb{N}}[+, \cdot, =]$
3. FOL$_{\mathbb{R}}[+, \cdot, =, <]$
4. FOL$_{\mathbb{Q}}[+, \cdot, =]$
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- ? $\mathrm{FOL}_{\mathbb{R}}[+,\cdot,\exp,=,<]$ — unknown
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?	FOL$_{\mathbb{R}}[+, \cdot, \exp, =, <]$	unknown
×	FOL$_{\mathbb{R}}[+, \cdot, \sin, =, <]$ $\sin x = 0$	not semidecidable [Richardson'68]

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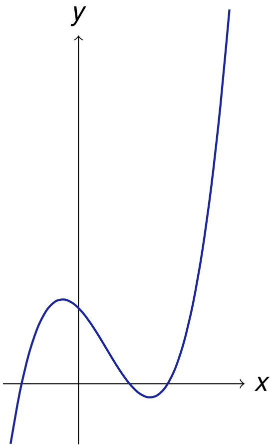
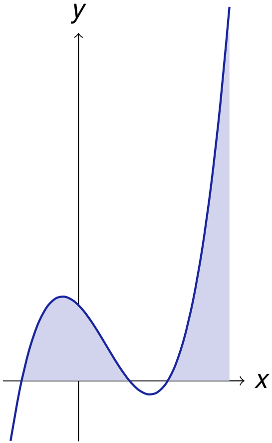
Quantifier Elimination $\rightsquigarrow$ Projection



$$F \equiv \exists y \, (y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$



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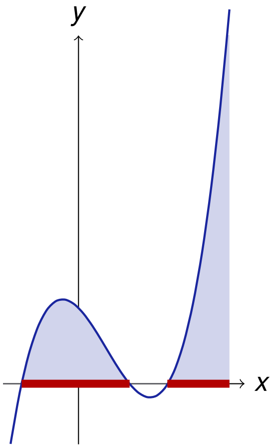

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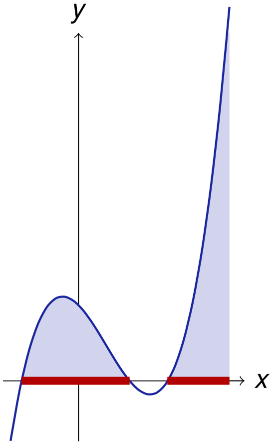

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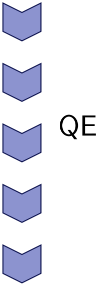

$$\mathrm{QE}(F) \equiv -0.75 < x \land x < 0.68 \lor x > 1.18$$

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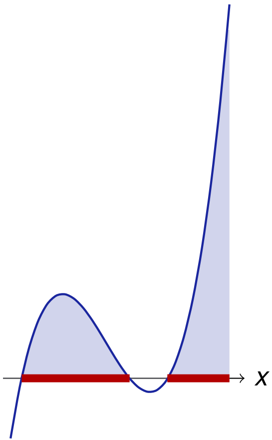
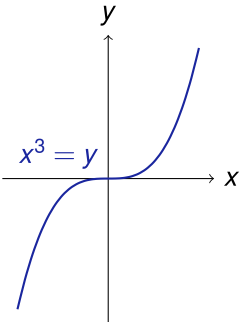


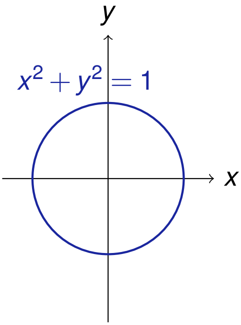
QE

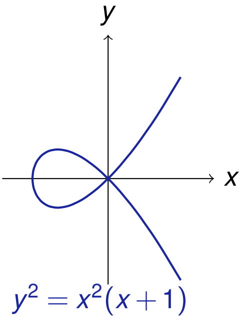
$$\mathrm{QE}(F) \equiv -0.75 < x \land x < 0.68 \lor x > 1.18$$

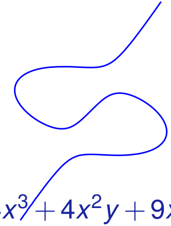
If all but one variable has fixed value: Finite union of intervals.

Univariate polynomials have finitely many roots. Signs change finitely often.





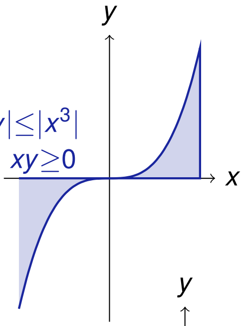


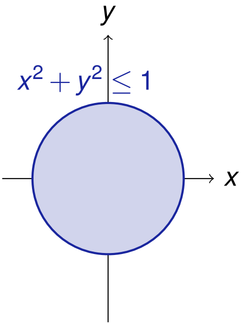


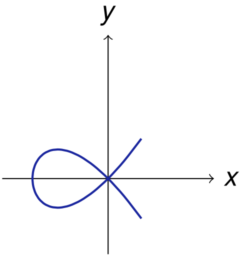
$z = x^2 - y^2$

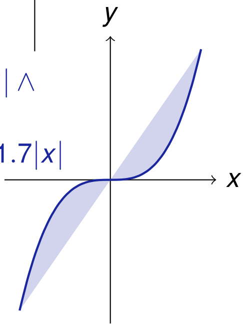
$4x^3 + 4x^2y + 9xy^2 - 9y^3 - 36x + 36y = 0$

Algebraic variety: defined by conjunction of polynomial equations


$|y| \leq |x^3|$
$\land\ xy \geq 0$
y
x


$x^2 + y^2 \leq 1$
y
x


y
x
$y^2 = x^2(x+1)$
$\land\ x \leq 0.44$


$|y| \geq |x^3| \land$
$xy \geq 0$
$\land\ |y| \leq 1.7|x|$
y
x

$z = x^2 - y^2$

Quantifier Elimination in Real Arithmetic

Theorem (Tarski'31)

First-order logic of real arithmetic is decidable since it admits quantifier elimination, i.e., for each formula P, compute quantifier-free formula $\mathrm{QE}(P)$ that is equivalent, i.e., $P \leftrightarrow \mathrm{QE}(P)$ is valid.

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE for $\mathbb{R}$ is doubly exponential in the number n of quantifier (alternations).

$$2^{2^{O(n)}}$$

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Answer even for one *free* variable and only linear polynomials

Quantifier Elimination Examples

$$\begin{aligned}
\text{QE}(\exists x\,(2x^2 + c \leq 5)) &\equiv \\
\text{QE}(\forall c\,\exists x\,(2x^2 + c \leq 5)) &\equiv
\end{aligned}$$

$$\begin{aligned}
\text{QE}(\exists x\,(a = b + x^2)) &\equiv \\
\text{QE}(\exists x\,(x^2 = 2)) &\equiv \\
\text{QE}(\exists x\,(x^2 = 2 \land y = x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

$$\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\leq 5)) &\equiv c\leq 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\leq 5)) &\equiv
\end{aligned}$$

$$\begin{aligned}
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\wedge y=x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

$$
\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\le 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2+c\le 5))
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
\end{aligned}
$$

Quantifier Elimination Examples

$$\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\le 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2+c\le 5)) \equiv \mathrm{QE}(\forall c\,(c\le 5))
\end{aligned}$$

$$\begin{aligned}
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\wedge y=x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

$$\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2 + c \le 5)) &\equiv c \le 5 \\
\mathrm{QE}(\forall c\,\exists x\,(2x^2 + c \le 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2 + c \le 5)) \equiv \mathrm{QE}(\forall c\,(c \le 5)) \\
&\equiv -100 \le 5 \land 5 \le 5 \land 100 \le 5 \\
\mathrm{QE}(\exists x\,(a = b + x^2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2 \land y = x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

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\mathrm{QE}(\exists x\,(2x^2+c\leq 5)) &\equiv c\leq 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\leq 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2+c\leq 5)) \equiv \mathrm{QE}(\forall c\,(c\leq 5))\\
&\equiv -100\leq 5\land 5\leq 5\land 100\leq 5\equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
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Quantifier Elimination Examples

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&\equiv -100\le 5\land 5\le 5\land 100\le 5 \equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
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Quantifier Elimination Examples

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Quantifier Elimination Examples

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&\equiv -100\le 5\land 5\le 5\land 100\le 5 \equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv \mathit{true}\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv y=\pm\sqrt{2}
\end{aligned}
$$

Quantifier Elimination Examples

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\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
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&\equiv -100\le 5\land 5\le 5\land 100\le 5\equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv \mathit{true}\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv y=\pm\sqrt{2}\equiv y^2=2
\end{aligned}
$$

Framework: Logical Normalization for QE

$$\begin{aligned}
\mathrm{QE}(A \wedge B) &\equiv \\
\mathrm{QE}(A \vee B) &\equiv \\
\mathrm{QE}(\neg A) &\equiv \\
\mathrm{QE}(\forall x\, A) &\equiv \\
\mathrm{QE}(\exists x\, A) &\equiv && A \text{ has quantifiers}
\end{aligned}$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B)\\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B)\\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A)\\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A)\\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers}
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
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\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
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\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A)
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A) \\
\mathrm{QE}(\exists x\, (A \land (B \lor C))) &\equiv \\
\mathrm{QE}(\exists x\, ((A \lor B) \land C)) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
QE(A \land B) &\equiv QE(A) \land QE(B) \\
QE(A \lor B) &\equiv QE(A) \lor QE(B) \\
QE(\neg A) &\equiv \neg QE(A) \\
QE(\forall x\, A) &\equiv QE(\neg \exists x\, \neg A) \\
QE(\exists x\, A) &\equiv QE(\exists x\, QE(A)) && A \text{ has quantifiers} \\
QE(\exists x\, (A \lor B)) &\equiv QE(\exists x\, A) \lor QE(\exists x\, B) \\
QE(\exists x\, \neg(A \land B)) &\equiv QE(\exists x\, (\neg A \lor \neg B)) \\
QE(\exists x\, \neg(A \lor B)) &\equiv QE(\exists x\, (\neg A \land \neg B)) \\
QE(\exists x\, \neg\neg A) &\equiv QE(\exists x\, A) \\
QE(\exists x\, (A \land (B \lor C))) &\equiv QE(\exists x\, ((A \land B) \lor (A \land C))) && \text{expensive} \\
QE(\exists x\, ((A \lor B) \land C)) &\equiv QE(\exists x\, ((A \land C) \lor (B \land C))) && \text{expensive}
\end{aligned}
$$

Framework: Logical Normalization for QE

Normal Form $\mathrm{QE}(\exists x\,(A_1 \land \ldots \land A_k))$ with atomic A_i

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\,\mathrm{QE}(A) \\
\mathrm{QE}(\forall x\,A) &\equiv \mathrm{QE}(\neg\exists x\,\neg A) \\
\mathrm{QE}(\exists x\,A) &\equiv \mathrm{QE}(\exists x\,\mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\,(A \lor B)) &\equiv \mathrm{QE}(\exists x\,A) \lor \mathrm{QE}(\exists x\,B) \\
\mathrm{QE}(\exists x\,\neg(A \land B)) &\equiv \mathrm{QE}(\exists x\,(\neg A \lor \neg B)) \\
\mathrm{QE}(\exists x\,\neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\,(\neg A \land \neg B)) \\
\mathrm{QE}(\exists x\,\neg\neg A) &\equiv \mathrm{QE}(\exists x\,A) \\
\mathrm{QE}(\exists x\,(A \land (B \lor C))) &\equiv \mathrm{QE}(\exists x\,((A \land B) \lor (A \land C))) && \text{expensive} \\
\mathrm{QE}(\exists x\,((A \lor B) \land C)) &\equiv \mathrm{QE}(\exists x\,((A \land C) \lor (B \land C))) && \text{expensive}
\end{aligned}
$$

Framework: Arithmetical Normalization for QE

Normal Form $\mathrm{QE}(\exists x\, (p_1 \sim_i 0 \land \ldots \land p_k \sim_k 0))$ and $\sim_i \in \{>, =, \geq, \neq\}$

$$p = q \equiv p - q = 0$$
$$p \geq q \equiv p - q \geq 0$$
$$p > q \equiv p - q > 0$$
$$p \neq q \equiv p - q \neq 0$$
$$p \leq q \equiv q - p \geq 0$$
$$p < q \equiv q - p > 0$$
$$\neg(p \geq q) \equiv p < q$$
$$\neg(p > q) \equiv p \leq q$$
$$\neg(p = q) \equiv p \neq q$$
$$\neg(p \neq q) \equiv p = q$$

Quantifier Elimination by Virtual Substitution

Virtual Substitution

$$\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \land F_x^t$$

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Scalability requires simplifier for intermediate results

Quantifier Elimination by Virtual Substitution

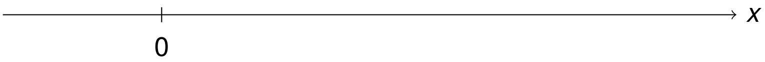
Virtual Substitution

Quantifier $\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \wedge F_x^t$ Quantifier-free

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Scalability requires simplifier for intermediate results

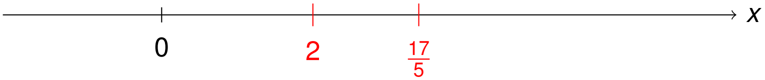
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x (x > 2 \land x < \frac{17}{5})$$

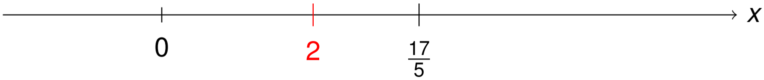
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x \left(x > 2 \land x < \frac{17}{5}\right)$$

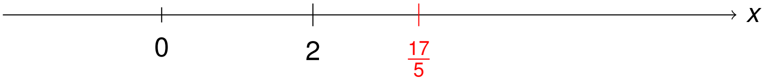
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

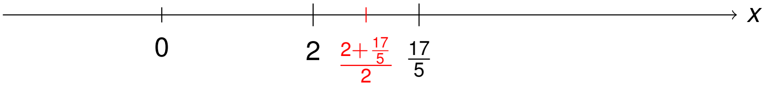
$$
\begin{aligned}
& \exists x (x > 2 \land x < \tfrac{17}{5}) \\
\equiv \quad & (2 > 2 \land 2 < \tfrac{17}{5}) \qquad \text{boundary case "}x = 2\text{"}
\end{aligned}
$$

Naïve Virtual Substitution by Example



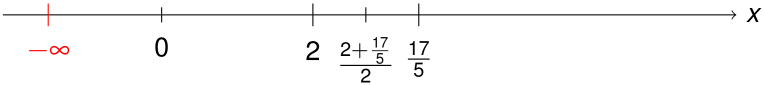
Can we get rid of the quantifier without changing the semantics?

$$\begin{array}{lll} & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\ \equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "} x = 2 \text{"} \\ \lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "} x = \frac{17}{5} \text{"} \end{array}$$



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “} x = 2 \text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “} x = \frac{17}{5} \text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “} x = \frac{2+\frac{17}{5}}{2} \text{”}
\end{array}
$$



Can we get rid of the quantifier without changing the semantics?

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\begin{array}{lll}
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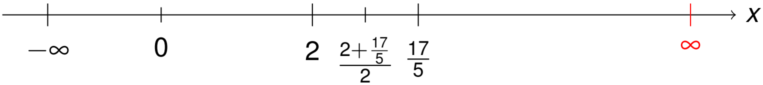
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”}
\end{array}
$$



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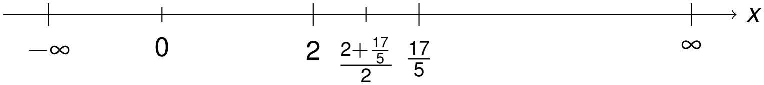
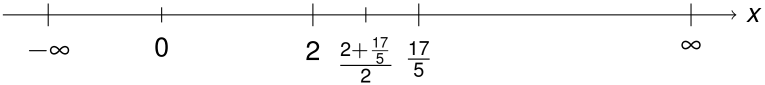
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”}
\end{array}$$



Can we get rid of the quantifier without changing the semantics?

$$\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "} x = 2 \text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "} x = \frac{17}{5} \text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "} x = \frac{2+\frac{17}{5}}{2} \text{"} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case "} x = -\infty \text{"} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case "} x = \infty \text{"} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}$$

Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

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\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

- ∞ is not in $\mathrm{FOL}_{\mathbb{R}}$
- Interior points aren't always terms in $\mathrm{FOL}_{\mathbb{R}}$ if nonlinear
- Substituting them into formulas requires attention

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \;\leftrightarrow$$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \;\leftrightarrow\; F_x^{-c/b}$$

Linear solution

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}$$

Don't divide by 0

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$b \neq 0 \rightarrow \left(\exists x\, (bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}\right)$$

Only actually linear solution if $b \neq 0$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \leftrightarrow b \neq 0 \wedge F_x^{-c/b}\big) \quad \text{if } x \notin b, c$$

Only linear if no x in b, c

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \left(\exists x \, (bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}\right) \quad \textit{if } x \notin b, c$$

Conditional equivalence, so conditions may need to be checked or case-split

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}\big) \quad \textit{if } x \notin b, c$$

Lemma (Uniform substitution of linear equations)

The linear equation axiom is sound (b, c are arity 0 function symbols):

$$\exists lin \;\; b \neq 0 \rightarrow \big(\exists x\,(b \cdot x + c = 0 \land q(x)) \leftrightarrow q(-c/b)\big)$$

$$\exists x\,(\underbrace{(y^2+4)}_{b} \cdot x + \underbrace{(yz-1)}_{c} = 0 \land x^3 + x \geq 0) \leftrightarrow \left(-\frac{yz-1}{y^2+4}\right)^3 + \left(-\frac{yz-1}{y^2+4}\right) \geq 0$$

Outline

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$F_x^{(-b+\sqrt{b^2-4ac})/(2a)}$$

Quadratic solution

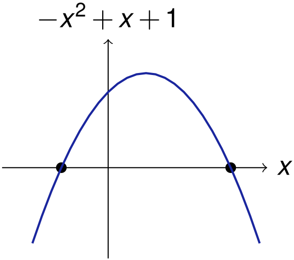
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$\left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Or negative square root solution



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2+bx+c=0 \land F) \leftrightarrow$$

$$a \neq 0 \land \qquad \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Don't divide by 0

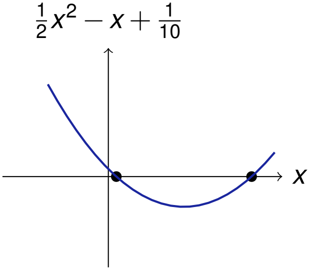
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x \, (ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Real solution if $\sqrt{\cdot}$ exists by discriminant



Quadratic Virtual Substitution

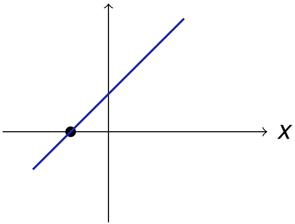
Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2+bx+c=0 \land F) \leftrightarrow$$

$$a=0 \land b\neq 0 \land F_x^{-c/b}$$

$$\lor\ a\neq 0 \land b^2-4ac\geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Instead linear solution if $a=0$ (may case-split)



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Theorem (Virtual Substitution: Quadratic Equation)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
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Only equivalent if not all 0 which gives trivial equation (else use F)

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
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Only linear or quadratic if no x in a, b, c

Quadratic Virtual Substitution

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$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
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1. Quantifier-free equivalent

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
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1. Quantifier-free equivalent
2. Just not a formula ...

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow a = 0 \land b \neq 0 \land F_x^{-c/b} \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

1. Quantifier-free equivalent
2. Just not a formula . . .
3. $(-b + \sqrt{b^2 - 4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$

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3. $(-b+\sqrt{b^2-4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$
4. Virtual substitution $F_{\bar{x}}^{(a+b\sqrt{c})/d}$ acts as if it were to substitute $(a+b\sqrt{c})/d$ for x in F

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5. $\exists r \, (r^2 = c)$ would do it for $\sqrt{c}$

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Square Root Algebra

Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\bar{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=}$$

Convention: On this slide c' is not a derivative but just another name ...

Square Root Algebra

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Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\bar{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=} p((a+b\sqrt{c})/d)$$

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$$p_{\bar{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=} p((a+b\sqrt{c})/d) \quad \text{by algebraic evaluation of } +, \cdot$$

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$\sqrt{c}$-algebra

Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a, b, c, d \in \mathbb{Q}[x_1, .., x_n]$:

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') =$$

$$((a+b\sqrt{c})/d) \cdot ((a'+b'\sqrt{c})/d') =$$

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Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a, b, c, d \in \mathbb{Q}[x_1, .., x_n]$:
where $c \geq 0, d, d' \neq 0$

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') = ((ad'+da') + (bd'+db')\sqrt{c})/(dd')$$
$$((a+b\sqrt{c})/d) \cdot ((a'+b'\sqrt{c})/d') = ((aa'+bb'c) + (ab'+ba')\sqrt{c})/(dd')$$

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Virtual $\sqrt{\cdot}$ Substitution

Virtual Substitution into Comparisons

Virtually substitute $(a+b\sqrt{c})/d$ into a comparison $p \sim 0$:

$$(p \sim 0)_{\bar{x}}^{(a+b\sqrt{c})/d} \equiv$$

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$\sqrt{c}$-comparisons $\quad d \neq 0 \land c \geq 0$

$$(a+0\sqrt{c})/d = 0 \equiv$$

$$(a+0\sqrt{c})/d \leq 0 \equiv$$

$$(a+0\sqrt{c})/d < 0 \equiv$$

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$$(a+b\sqrt{c})/d \leq 0 \equiv$$

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$$(a+0\sqrt{c})/d = 0 \equiv a = 0$$

$$(a+0\sqrt{c})/d \leq 0 \equiv ad \leq 0$$

$$(a+0\sqrt{c})/d < 0 \equiv ad < 0$$

$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \land a^2 - b^2 c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \land a^2 - b^2 c \geq 0 \lor bd \leq 0 \land a^2 - b^2 c \leq 0$$

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Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic → $F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$ ← $\mathrm{FOL}_{\mathbb{R}}$

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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$ FOL$_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \text{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$$

Example: Quadratic Curiosity

$$a \neq 0 \rightarrow (\exists x\,(ax^2+bx+c=0 \land ax^2+bx+c \leq 0) \leftrightarrow b^2-4ac \geq 0 \land \mathit{true})$$

$$
\begin{aligned}
&(ax^2+bx+c)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}\\
=&a((-b+\sqrt{b^2-4ac})/(2a))^2+b((-b+\sqrt{b^2-4ac})/(2a))+c\\
=&a((b^2+b^2-4ac+(-b-b)\sqrt{b^2-4ac})/(4a^2))+(-b^2+b\sqrt{b^2-4ac})/(2a)+c\\
=&(ab^2+ab^2-4a^2c+(-ab-ab)\sqrt{b^2-4ac})/(4a^2)+(-b^2+2ac+b\sqrt{b^2-4ac})/(2a)\\
=&((ab^2+ab^2-4a^2c)2a+(-b^2+2ac)4a^2+((-ab-ab)2a+b4a^2)\sqrt{b^2-4ac})/(8a^3)\\
=&(\cancel{2a^2b^2}+\cancel{2a^2b^2}-\cancel{8a^3c}-\cancel{4a^2b^2}+\cancel{8a^3c}+(-\cancel{2a^2b}-\cancel{2a^2b}+\cancel{4a^2b})\sqrt{b^2-4ac})/(8a^3)\\
=&(0+0\sqrt{b^2-4ac})/(8a^3)=(0+0\sqrt{..})/1=0
\end{aligned}
$$

$$(ax^2+bx+c=0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv ((0+0\sqrt{..})/1=0) \equiv (0\cdot 1=0) \equiv \mathit{true}$$

$$(ax^2+bx+c\leq 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv (\underbrace{(0+0\sqrt{..})/1}_{0} \leq 0) \equiv (0\cdot 1 \leq 0) \equiv \mathit{true}$$

Example: Nonnegative Roots of Quadratic Polynomials

$$
\begin{aligned}
a \neq 0 \rightarrow \big(\exists x\,(ax^2 + bx + c = 0 \land x \geq 0)\\
\leftrightarrow b^2 - 4ac \geq 0 \land \big(2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0\\
\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)
\end{aligned}
$$

$$
\begin{aligned}
-(-b + \sqrt{b^2 - 4ac})/(2a) &= ((-1 + 0\sqrt{b^2 - 4ac})/1) \cdot ((-b + \sqrt{b^2 - 4ac})/(2a)\\
&= (b - \sqrt{b^2 - 4ac})/(2a)
\end{aligned}
$$

$$
\begin{aligned}
& (-x \leq 0)_{\bar{x}}^{(b-\sqrt{b^2-4ac})/(2a)}\\
\equiv\ & b2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \geq 0 \lor -1 \cdot 2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \leq 0\\
\equiv\ & 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0\\
& (-x \leq 0)_{\bar{x}}^{(b+\sqrt{b^2-4ac})/(2a)}\\
\equiv\ & b2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \geq 0 \lor 1 \cdot 2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \leq 0\\
\equiv\ & 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0
\end{aligned}
$$

Outline

$\sqrt{\cdot}$ Square Root Algebra

Virtual Substitution of $(a+b\sqrt{c})/d$ into Comparisons

$$(p \sim 0)_{\bar{x}}^{(a+b\sqrt{c})/d} \equiv (p_{\bar{x}}^{(a+b\sqrt{c})/d} \sim 0) \quad \text{accordingly for } \land, \lor, \ldots$$

$\sqrt{c}$-algebra $\quad d \neq 0 \land c \geq 0$

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') = ((ad'+da') + (bd'+db')\sqrt{c})/(dd')$$

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$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \land a^2 - b^2c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \land a^2 - b^2c \geq 0 \lor bd \leq 0 \land a^2 - b^2c \leq 0$$

$$(a+b\sqrt{c})/d < 0 \equiv ad < 0 \land a^2 - b^2c > 0$$
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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \quad$ $FOL_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \text{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$$

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