Recursion 5

"To understand recursion, you must first understand recursion"

Recursion

- A recursive function is a function that is defined in terms of itself.
- Every recursive definition must have a base case that is not recursive.
  - The non-recursive nature of the base case allows us to then solve previous recursive steps.
- There can be more than one base case.

Factorial

- n!   = n * (n-1) * (n-2) * ... * 2 * 1 for n > 0
        = 1 for n = 0
- But, since (n-1)! = (n-1) * (n-2) * 2 * 1, we can use recursion to define the factorial function:
  n!   = n * (n-1)!   for n > 0
        = 1   for n = 0 (base case)
- Example:
  4! = 4*3! = 4*(3*2!) = 4*(3*(2*1!)) = 4*(3*(2*(1*0!)))
        = 4*(3*(2*(1*1))) = 4*(3*(2*1)) = 4*(3*2) = 4*6 = 24

Factorial in Java

```
public static int factorial(int n) {
  // Precondition: n >= 0
  int result;
  if (n == 0)
      result = 1;
  else
      result = n * factorial(n-1);
  return result;
 }
```

Improved Factorial

```
public static int factorial(int n) {
  if (n < 0)
      throw new IllegalArgumentException();
  int result;
  if (n == 0)
      result = 1;
  else
      result = n * factorial(n-1);
  return result;
 }
```


Another Factorial

```
public static int factorial(int n) {
  if (n < 0)
      throw new IllegalArgumentException();
  return fact(n, 1);
}
private static int fact(int n, int total) {
  if (n == 0)
      return total;
  return fact(n-1, n*total);   ← tail recursive
 }
```

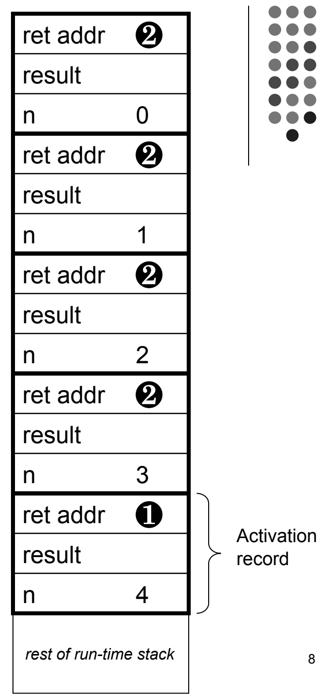
Tail Recursion

A tail recursive method can always be converted into an iterative one. Which is more efficient?

```
public static int factorial(int n) {
  // Precondition: n >= 0
  int result = 1;
  for (int k = 1; k <= n; k++)
      result *= k;
  return result;
 }
```


Recursion and the Run-Time Stack

```
...
❶ int x = factorial(4);
...

public static int factorial(int n) {
   // Precondition: n >= 0
   int result;
   if (n == 0)
      result = 1;
   else
   ❷ result = n * factorial(n-1);
   return result;
}
```



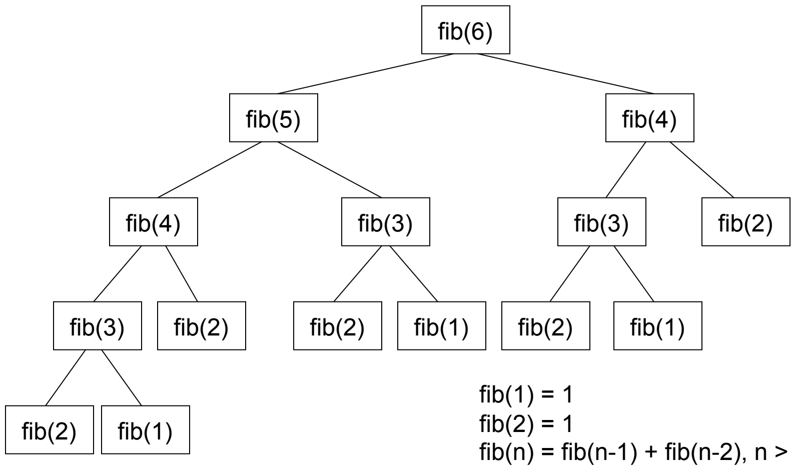
Fibonacci Numbers

Fibonacci Numbers in Java

$fib_n = fib_{n-1} + fib_{n-2}$ for n > 2
$fib_2 = 1$
$fib_1 = 1$

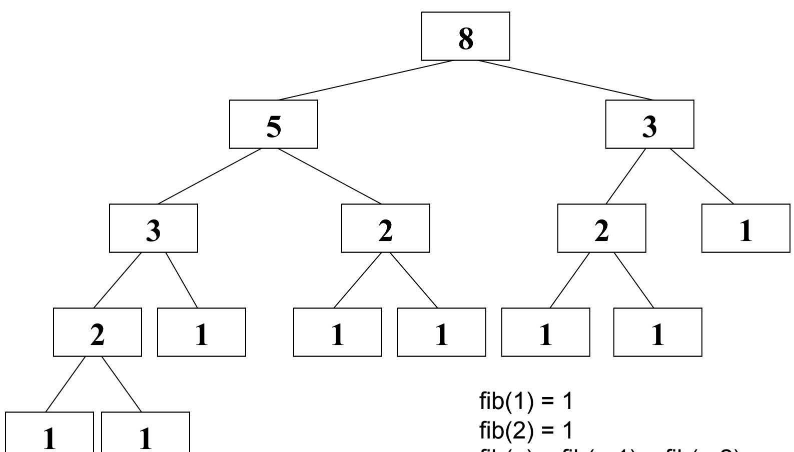
```
public static int fib(int n) {
  // Precondition: n > 0
  if (n <= 2)
   return 1;
  else
   return fib(n-1) + fib(n-2);
}
```

Fibonacci Numbers



fib(1) = 1
fib(2) = 1
fib(n) = fib(n-1) + fib(n-2), n > 2

Fibonacci Numbers



fib(1) = 1
fib(2) = 1
fib(n) = fib(n-1) + fib(n-2), n > 2

Fibonacci Numbers Iteratively

```
public static int fib(int n) {
   // Precondition: n > 0
   int fibprev2 = 0;
   int fibprev = 1;
   int fibcurr = 1;
   for (int i = 3; i <= n; i++) {
     fibprev2 = fibprev;
     fibprev = fibcurr;
     fibcurr = fibprev + fibprev2;
   }
   return fibcurr;
}
```

Order of complexity:

Iterative: O(n)

Recursive: $O(2^n)$

if n is too high, this can result in `StackOverflowError`

Greatest Common Divisor

```
public static int gcd(int m, int n) {
   // Precondition: m > 0, n > 0
   if (m % n == 0)
     return n;
   else
     return gcd(n, m % n);
}
```

String length

```
public static int length(String s)
{
  // Precondition: s != null

  if (________________________________)
    return 0;
  else
    return ________________________________________;
}
```


Linear Search

```
private static int search(int[] a, int target, int index)
{
  // Search array a for target starting at index

  if (________________________________)
    return index;
  else if (________________________________)
    return -1;
  else
    return ________________________________________;
}
```

Use of wrapper method

```
public static int search(int[] a, int target)
{
  return search(a, target, 0);       // prev. slide
}
```

User calls this method to search entire array.

This method calls the recursive method to start the search with index 0.

Exponentiation (x^n)

$x^n = x * \underbrace{x * x * ... * x}_{n-1} = x * x^{n-1}$, for n > 0.

```
public static double power(double x, int n)
{
  // Precondition: n >= 0



}
```

What's wrong?

We can also say (recursively) that $x^n = x^{n/2} * x^{n/2}$.

What's wrong with the following recursive step for exponentiation?

```
return power(x, n/2) * power(x, n/2);
```


Linked Lists Recursively

Assume we have a singly-linked list as defined below:

```
public class SinglyLinkedList {
  private Node<E> head;

  private static class Node<E> {
      ...
  }
}
```

Finding the size of the list

```
public int size() {           // wrapper method
  size(head);
}
private int size(Node<E> nodeRef) {
  // Find size of list starting at nodeRef
  if (nodeRef == null)
      return 0;
  else
      return 1 + size(nodeRef.next);
}
```


Adding a new node to the end of the list

```
public void add(E element) {  // wrapper method
  if (head == null)
   head = new Node<E>(element);
  else
   add(head, element);
}
private void add(Node<E> nodeRef, E element) {
  if (nodeRef.next == null)
   nodeRef.next = new Node<E>(element);
  else
   add(nodeRef.next, element);
}
```

Add `element` to the end of the list that starts at the node referenced by `nodeRef`

What's wrong?

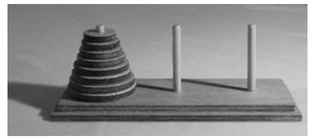
```
public void add(E element) {  // wrapper method
  add(head, element);
}

private void add(Node<E> nodeRef, E element) {
  if (nodeRef == null)
   nodeRef = new Node<E>(element);
  else
   add(nodeRef.next, element);
}
```

Add `element` to the end of the list that starts at the node referenced by `nodeRef`

Counting number of matches

```
public int count(E element) {       // wrapper method
  return count(head, element);
}
private int count(Node<E> nodeRef, E element) {
  // Returns number of matches of element in linked list
  // starting from node referenced by nodeRef







}
```

Towers of Hanoi

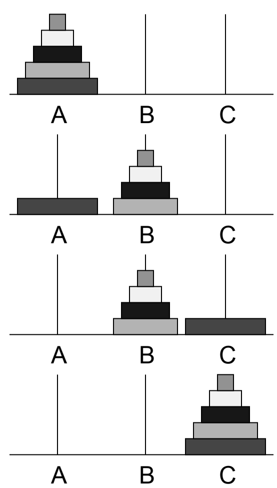


Towers of Hanoi with 8 discs.

- A puzzle invented by French mathematician Edouard Lucas in 1883.
- At a monastery far away, monks were led to a courtyard with three pegs and 64 discs stacked on one peg in size order.
  - Monks are only allowed to move one disc at a time from one peg to another.
  - Monks may not put a larger disc on top of a smaller disc at any time.
- The goal of the monks was to move all 64 discs from the leftmost peg to the rightmost peg.
- According to the legend, the world would end when the monks finished their work.

Recursive Solution

example: N=5



Move N discs from peg A to peg C (Let B represent the extra peg.)

recursive

a. If N > 1, move N-1 discs
from peg A to peg B.

b. Move 1 disc
from peg A to peg C.

c. If N > 1, move N-1 discs
from peg B to peg C.

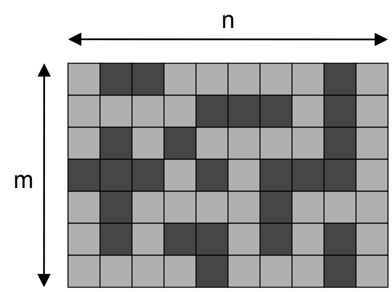
Backtracking

- Algorithmic technique to search a large problem space for a solution.
- At each point in the search, we have a number of choices.
- As we pick a choice and proceed on, we may hit a "dead end".
- Backtracking involves "backing up" to the most recent choice and choosing another possible choice to follow.

Finding a path in a maze

- Let a maze be represented as an $m \times n$ array with two colors.
  - All cells that are part of the maze are painted in one color (GREEN).
  - All cells that are not part of the maze are painted in another color (RED).
- The entry point in the maze is (0,0).
- The exit point in the maze is (m-1, n-1).

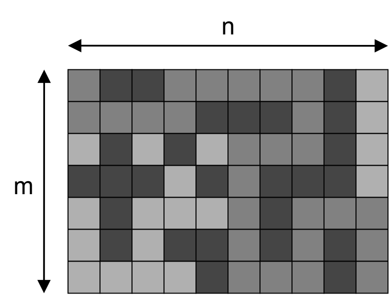
Finding a path in a maze



Is there a path from cell x,y to the exit?

Initially, x = 0 and y = 0.

Finding a path in a maze



If there is a path, mark each node along the path in blue.

Finding a path in a maze
Base Cases

Is there a path from (x,y) to (m-1, n-1)?

- If (x,y) is outside the maze boundaries, answer NO.
- If (x,y) is a RED cell, answer NO.
- If (x,y) has already been visited, answer NO.
- If (x,y) is (m-1, n-1),
  color this cell in blue and answer YES.

How do we know if a cell has already been visited?

- Color the cell in yellow.

Finding a path in a maze
Recursive Step

Is there a path from (x,y) to (m-1, n-1)?
If none of the base cases apply...

- Color cell (x,y) in blue.
- For each neighbor of (x,y), ← recursive
  - If there is a path from the neighbor to (m-1,n-1), answer YES.
- Otherwise, recolor cell (x,y) with yellow
  (i.e. visited but not part of the path)
  and answer NO.